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"Q: Is space curved?
A: It is in Leeds."
— Doktor Avalanche

University of Alberta

BLACK HOLE SOLUTIONS OF FIVE-DIMENSIONAL RELATIVITY WITH NEW HORIZON TOPOLOGIES

by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Master of Science**

in

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Black hole solutions of five-dimensional relativity with new horizon topologies** submitted by Coire D. Cadeau in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematical Physics**.

To Dale, with love.

Abstract

Two new families of solutions to the vacuum Einstein equations with negative cosmological constant are presented. The time-symmetric hypersurfaces of these solutions are foliated by homogeneous three-dimensional manifolds generated by isometries of the spacetime. The new solutions are static black holes with event horizons modeled on the three-dimensional geometries nilgeometry and solvegeometry. The topology and compactification of the respective horizon manifolds is explicitly described. Known solutions are exhibited for each of the remaining three-dimensional geometries, except for the case of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, which remains open.

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This thesis was typeset using LaTeX, together with the `mathrsfs` font and the `amsfonts` package.

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Chapter 1

Introduction

Since the early 1990s, it has become clear that in four-dimensional general relativity there are black hole solutions possessing event horizons that are not spheres. For example, Lemos [27] extended a three-dimensional black hole to obtain a four-dimensional black hole whose event horizon was foliated by cylinders. Later work by Åminneborg *et al.* [1] showed that it was also possible to have a black hole solution in four dimensions whose event horizon was a Riemann surface of arbitrary genus.

Recently, five-dimensional relativity has been of interest in physics owing to connections with string theory via the conjectured AdS/CFT correspondence [28], and also because of current cosmological ideas where the universe is thought of as a hypersurface in a larger overall spacetime (the "braneworld" scenario, discussed by Randall and Sundrum [34], Krauss [25], and Ida [19], among others). However, there is little known about what topologies are allowed for five-dimensional spacetimes.

In 1982, William Thurston [40] stated his now famous geometrisation conjecture. This conjecture holds that every compact three-dimensional manifold can be decomposed in a unique way such that each of the resulting pieces are covered by one of eight three-dimensional "model geometries." Fagundes [12] pointed out that these model geometries admit homogeneous metrics nearly in correspondence with the well-known homogeneous Bianchi manifolds from cosmology. This conjecture, if it is true, would provide a route for classifying the manifolds that could serve as event horizons in a five-dimensional black hole.

With this conjecture in mind, we set out to determine whether one can find a static five-dimensional Einstein manifold with a black hole, whose event horizon is foliated by a compact manifold covered by a given model geometry. Aside from providing insight into the allowed topologies for five-dimensional black holes, the solutions themselves may also be physically interesting, as they could provide insight into positive mass theorems, or new

solutions with which to construct a braneworld.

In this thesis, two new five-dimensional black hole solutions are presented, with horizons modeled on nilgeometry and solvegeometry. With these two solutions, all model geometries now have a corresponding black hole solution, except for the case of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. It is not known whether there is a fundamental obstruction to obtaining a solution with this topology, or whether the lack of a solution resulted from adopting assumptions that were too special.

Chapter 2 begins with a short review of ideas from (semi-)Riemannian geometry, and then proceeds to discuss Thurston's idea of "model geometries." The less familiar cases of compact manifolds modeled on nilgeometry and solvegeometry are illustrated with explicit examples.

In chapter 3, the formalism of general relativity is reviewed. The Schwarzschild solution is used to demonstrate features of static black holes. This leads to a precise definition of "black hole" that Hawking proposed [17], but seems to fail for certain solutions that are nevertheless properly thought of as black objects. The chapter concludes with a brief discussion of black hole thermodynamics.

In chapter 4 a formal statement of the research problem is made, and both the known and new solutions satisfying our requirements are presented.

Chapter 2

Three-dimensional geometries

In this chapter, we begin by recalling some important notions from (semi-)Riemannian geometry. We will proceed to discuss Thurston's model geometries in three dimensions, and their link to Bianchi's classification of the nine allowed three-dimensional Lie groups familiar from cosmology. The chapter concludes with a statement and brief discussion of Thurston's geometrisation conjecture.

In this thesis, all manifolds M are considered to be smooth (C^∞) and Hausdorff. A Riemannian manifold M with metric $g : T_p(M) \times T_p(M) \to \mathbb{R}$ is written (M, g). The treatment of semi-Riemannian geometry follows the development of several texts listed in the bibliography; especially useful are O'Neill [32], and Petersen [33]. Thurston's model geometries are treated in Thurston [41] and an article by Scott [37]. A version of Section 2.4 and the included subsections are published in Cadeau and Woolgar [8].

2.1 Isometries of Riemannian manifolds

First recall some facts about (smooth) maps between manifolds, and the related induced maps on the tangent, cotangent, and tensor bundles. Consider a map $\phi : M \to N$ between two manifolds M and N. If f is a function on N, we can define a new function $\phi^* f$ on M called the "pull back" of f, as

$$\phi^* f(p) = f(\phi(p)). \tag{2.1}$$

This is a linear map of functions on N to functions on M.

Remembering the definition of a tangent vector as the directional derivative of a function, we can also obtain a linear map ϕ_* from $T_p(M)$ to $T_{\phi(p)}(N)$. If $X \in T_p(M)$,

$$X(\phi^* f)|_p = \phi_* X(f)|_{\phi(p)}. \tag{2.2}$$

Similarly we can also obtain a linear map ϕ^* of one-forms from $T^*_{\phi(p)}(N)$ to $T^*_p(M)$ by requiring that the contraction of vectors with one-forms is to be preserved under the maps. So if $\omega \in T^*_{\phi(p)}(N)$, we have

$$< \phi^*\omega, X > |_p =< \omega, \phi_* X > |_{\phi(p)}, \tag{2.3}$$

for arbitrary vectors $X \in T_p(M)$. These maps are extended in the obvious way to maps of contravariant and covariant tensors of higher rank.

Riemannian geometry can be thought of as the study of those properties of a Riemannian manifold which are preserved under an equivalence. The proper notion of equivalence for Riemannian manifolds is isometry [32]:

Definition 2.1 (isometry) Let (M, g) and (N, h) be Riemannian manifolds. An *isometry* from M to N is a diffeomorphism $\phi : M \to N$ that preserves the metric tensor, so that $\phi^* h = g$. More explicitly, if $X, Y \in T_p(M)$, then

$$g(X, Y)|_p = h(\phi_* X, \phi_* Y)|_{\phi(p)}. \tag{2.4}$$

Two manifolds are said to be *isometric* if there exists an isometry between them.

We shall usually consider isometries $\phi : M \to M$ of a space into itself. The isometries of a manifold possess a group structure: clearly the identity map is an isometry, the composition of isometries is an isometry, and the inverse map of an isometry is an isometry. Generally speaking, the largest group of isometries from a particular manifold to itself is the trivial one; if the isometry group is larger, the manifold is said to have a symmetry.

We also have the notion of homogeneity of a Riemannian manifold:

Definition 2.2 (homogeneous, transitive) A Riemannian manifold M is said to be *homogeneous* if its group G of isometries acts *transitively*. That is, for each $x, y \in M$, there exists $g \in G$ such that $gx = y$.

A homogeneous manifold has the property that every point "looks like" every other point, as for any two points on the manifold, there exists an isometry taking one point to the other. As isometries preserve the metric, any measurements involving length, distance, or angles that an observer might use to compare two different points on a manifold will be indistinguishable.

We will also need to consider the stabiliser, or isotropy subgroup of the group of isometries at a point.

Definition 2.3 (stabiliser, isotropy) Consider a Riemannian manifold M together with its isometry group, G. The *stabiliser* (or *isotropy*) (sub)group at a point $x \in M$, a Riemannian manifold, are those elements $g \in G$ such that $gx = x$. The stabiliser of x is denoted G_x.

2.2 The Levi-Civita connection and curvature

In this section, we will briefly present the notions of the Levi-Civita connection and curvature in Riemannian geometry. If $V, W \in \mathfrak{X}(M)$, the set of all smooth vector fields on M, we wish to obtain a new vector field $\nabla_V W$ on M such that the value at each point $x \in M$ is the vector rate of change of W in the V_x direction. In flat $n-$dimensional Euclidean space, $\mathbb{R}^n$, this is done simply by calculating the directional derivative of each of the n component functions describing W, $\nabla_V W = \sum_i V(W^i)\partial_i$. To establish an equivalent notion on an arbitrary Riemannian manifold, we define the notion of a *connection*:

Definition 2.4 ((Levi-Civita) connection) A *connection* ∇ on a smooth manifold M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:

1. $\nabla_{\alpha V+\beta W}X = \alpha\nabla_V X + \beta\nabla_W X$, where α, β are smooth real-valued functions on M.

2. ∇ is a derivation, that is, $\nabla_V(X+Y) = \nabla_V X+\nabla_V Y$, and $\nabla_V(fX) = (Vf)X+f\nabla_V X$.

We can define the *Levi-Civita connection* on a manifold by adding the following properties:

3. $[V, W] = \nabla_V W - \nabla_W V$, which says that ∇ is "torsion-free."

4. $Xg(V, W) = g(\nabla_X V, W) + g(V, \nabla_X W)$, which says that ∇ is "metric."

The Levi-Civita connection on a Riemannian manifold exists, and is unique.

If we enter into a co-ordinate system $x^1, \ldots, x^n$, we can write the connection in terms of the *Christoffel symbols*, Γ^k_{ij} which are defined by the equation

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k,$$

where we have used Einstein's summation convention. So, if $V = V^i(x)\partial_i, X = X^i(x)\partial_i$, then we have

$$\begin{aligned}\nabla_V X &= V^i\nabla_{\partial_i}(X^j\partial_j) \\ &= V^i\left(\frac{\partial X^k}{\partial x^i} + X^j\Gamma^k_{ij}\right)\partial_k.\end{aligned}$$

We also have that the Christoffel symbols can be computed via the metric according to the formula

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}\left(g_{jm,i} + g_{im,j} - g_{ij,m}\right), \tag{2.5}$$

where the comma indicates partial differentiation with respect to a co-ordinate.

In general, the covariant derivative ∇_X fails to commute, and this failure gives rise to the notion of curvature in Riemannian geometry.

Definition 2.5 (Riemannian curvature tensor) If M is a Riemannian manifold with Levi-Civita connection ∇, then the function $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R_{XY}Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

is a $(1,3)$ tensor field on M called the *Riemannian curvature tensor* of M.

In the $x^1, \ldots, x^n$ co-ordinate system, we can write the components of the Riemann curvature tensor as

$$R_{\partial_j \partial_i} \partial_k = R^l{}_{ijk} \partial_l,$$

which have the following expression in terms of the Christoffel symbols and their derivatives:

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{ak}\Gamma^a_{jl} - \Gamma^i_{al}\Gamma^a_{jk}. \tag{2.6}$$

To compute the Riemann curvature tensor for a given manifold, one can simply compute the Christoffel symbols from Equation 2.5, and then employ Equation 2.6. Another route to the Riemann curvature tensor is detailed in Misner *et al.* [31], and Ryan and Shepley [35], for example. In this formalism, one selects a basis of one-forms ω^μ, and writes the metric as

$$ds^2 = g_{\mu\nu}\omega^\mu \omega^\nu. \tag{2.7}$$

One proceeds by solving for the connection one-forms $\omega^\mu{}_\nu$, which are uniquely determined by the equations

$$\text{(Cartan's first equation)} \qquad 0 = d\omega^\mu + \omega^\mu{}_\nu \wedge \omega^\nu \tag{2.8}$$

$$\text{(Compatibility)} \qquad dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}. \tag{2.9}$$

From this, one obtains the curvature two-forms $\mathscr{R}^\mu{}_\nu$ via Cartan's second equation

$$\mathscr{R}^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu. \tag{2.10}$$

The correspondence between the components of the Riemann curvature tensor in the selected basis and the curvature two-forms is given by the formula

$$\mathscr{R}^\mu{}_\nu = R^\mu{}_{\nu|\alpha\beta|}\omega^\alpha \wedge \omega^\beta, \tag{2.11}$$

where the sum is restricted to $\alpha < \beta$.

Two important quantities are obtained via contractions of the Riemann tensor, the Ricci curvature tensor, and the Ricci scalar:

Definition 2.6 (Ricci curvature tensor, Ricci scalar) For a Riemannian manifold (M, g), if $e_1, \ldots, e_n \in T_p(M)$ is an orthonormal basis, the *Ricci curvature tensor* is defined as the $(0, 2)$ tensor Ric

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} g(e_i, e_i) g(R_{e_i X} Y, e_i).$$

or equivalently as the $(1, 1)$ tensor

$$\mathrm{Ric}(X) = \sum_{i=1}^{n} g(e_i, e_i) R_{X e_i} e_i.$$

The *Ricci scalar curvature* scal is the contraction of Ric:

$$\mathrm{scal} = \sum_{j=1}^{n} g(e_i, e_i) g(\mathrm{Ric}(e_j), e_j)$$

If $R^{\alpha}{}_{\beta\gamma\delta}$ are the components of the Riemann tensor in a co-ordinate system, then we denote the components of Ric by $R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta}$, and scal by $R = R^{\alpha}{}_{\alpha}$.

Finally, the last notion of curvature that we shall need in this chapter is that of sectional curvature:

Definition 2.7 (sectional curvature) For a Riemannian manifold (M, g), let Π be a two-dimensional subspace of $T_p(M)$, and let $X, Y \in T_p(M)$ be tangent vectors at p spanning Π (that is, $g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0$). Then the *sectional curvature* $K(X, Y)$ of Π is defined as

$$K(X, Y) = \frac{g(R_{XY} X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

One can show that the sectional curvature of a tangent plane is independent of the choice of basis for Π, so we can write $K(\Pi)$ for the sectional curvature. In dimensions 3 and lower, it can be shown that there is no difference in knowing the sectional curvatures versus the Ricci curvature tensor; each can be obtained from the other.

Sectional curvature leads to the definition of a "constant curvature" manifold:

Definition 2.8 (constant curvature) A Riemannian manifold (M, g) is said to be *constant curvature* if for every $p \in M$, for every 2-plane Π in $T_p(M)$, $K(\Pi) = k$, k a constant.

We shall also have occasion to refer to an "Einstein manifold." As we shall see, Einstein manifolds are important in relativity, as they are vacuum solutions to Einstein's field equations.

Definition 2.9 (Einstein manifold) A Riemannian manifold (M, g) is said to be *Einstein* if $\mathrm{Ric}(X, Y) = kg(X, Y)$, for k a constant, for every $X, Y \in T(M)$.

In three dimensions and lower, the set of Einstein manifolds is identically the set of constant curvature manifolds [33].

2.3 Killing vectors and isometries

In our discussion, we will have occasion to use Killing vectors and the isometries generated by these vectors. This section gives a physical illustration of what Killing vectors are, and how they can be thought of as the generators of isometries on a manifold. The discussion below is based on [31].

Suppose the metric $g_{\mu\nu}$ in some co-ordinate system is independent of the co-ordinate x^K, for a fixed index K. This is to say that $g_{\mu\nu,K} = 0$. Following [31], we develop a geometric interpretation of this co-ordinate independence of the metric. Take a curve defined by $x^\alpha = c^\alpha(\lambda)$, and let us translate it by ε in the K direction to form an equivalent curve:

$$x^\alpha = c^\alpha(\lambda) \text{ when } \alpha \neq K, \; x^K = c^K(\lambda) + \varepsilon \text{ otherwise.}$$

Let us compute the lengths of these two curves: if the original curve runs from $\lambda = \lambda_1$ to $\lambda = \lambda_2$ then we can write down the length as

$$L = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu} \left(\frac{dx^\mu}{d\lambda}\right) \left(\frac{dx^\nu}{d\lambda}\right)} d\lambda.$$

If we write the metric on the translated curve as a first order Taylor expansion about the original curve to first-order in ε, we can write the length of the displaced curve as

$$L(\varepsilon) = \int_{\lambda_1}^{\lambda_2} \sqrt{\left(g_{\mu\nu} + \varepsilon \frac{\partial g_{\mu\nu}}{\partial x^K}\right) \left(\frac{dx^\mu}{d\lambda}\right) \left(\frac{dx^\nu}{d\lambda}\right)} d\lambda,$$

keeping in mind that $g_{\mu\nu}$ is evaluated on the original curve at the same value of λ. From the co-ordinate independence of $g_{\mu\nu}$, we have that the derivative multiplying ε in the integrand is zero. This says that the length of the new curve (for an infinitesimal translation) is identical to the original length, or $dL/d\varepsilon = 0$.

The vector

$$\vec{\xi} = \vec{e}_K$$

tells us the direction in which to translate points on a curve so that length is preserved. Such a vector is called a *Killing vector*, and it satisfies *Killing's equation*:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0. \tag{2.12}$$

We can establish Killing's equation by going into a preferred co-ordinate system where $\xi^\mu = \delta^\mu_K$. If a semicolon represents covariant differentiation with respect to a co-ordinate basis vector, we write the covariant derivative of ξ_μ as:

$$\xi_{\mu;\nu} = g_{\mu\alpha} \xi^\alpha{}_{;\nu} \quad = \quad g_{\mu\alpha} \left(\xi^\alpha{}_{,\nu} + \Gamma^\alpha_{\nu\sigma} \xi^\sigma\right)$$

$$
\begin{aligned}
\text{(using that } \xi^\alpha \text{ is the Kronecker here)} &= g_{\mu\alpha}\Gamma^\alpha_{\nu K} \\
&= \frac{1}{2} g_{\mu\alpha} g^{\alpha\beta}(g_{\beta\nu,K} + g_{\beta K,\nu} - g_{\nu K,\beta}) \\
\text{(metric is independent of } x^K\text{)} &= \frac{1}{2} g^\beta{}_\mu (g_{\beta K,\nu} - g_{\nu K,\beta}) \\
(g^\beta{}_\mu \text{is the Kronecker}) &= \frac{1}{2}(g_{\mu K,\nu} - g_{\nu K,\mu}),
\end{aligned}
$$

and this equation is anti-symmetric on interchange of μ and ν, which is what Equation 2.12 says. Because Equation 2.12 is in covariant form, that we have established it in one co-ordinate system means that it will hold in any co-ordinate system.

Because the length of curves under an infinitesimal translation in the direction of a Killing vector is unchanged, it follows easily that the distance between any two points on a curve is also preserved under this translation. The geometry of the manifold is therefore left completely unchanged by translations of all points under $\varepsilon\vec{\xi}$, and one calls the Killing vector $\vec{\xi}$ the *generator* of an isometry of the spacetime. The curves that are tangent to $\vec{\xi}$ are called *trajectories of the isometry.*

2.4 Model geometries

When studying geometry, there are several related viewpoints one may work with. For example, one can follow the classical model of Euclid and study the notions of points, lines, angles, and planes. Alternatively, one can study a space equipped with a metric, like a Riemannian manifold. Lastly, one can adopt Klein's picture and study a space equipped with a congruence. Thurston's [41] idea was that taken by themselves, each of these approaches have inherent weaknesses: Euclid's approach suffers because the requisite notions are not always easy, or possible, to formulate. Studying the spaces equipped with a metric leads to distinguishing spaces that we might wish to consider as geometrically equivalent: for example, simply scaling the metric by a constant factor leads, strictly speaking, to a different metric space. Similarly, Klein's approach would also lead to distinguishing geometries that we do not wish to distinguish; for example, we could consider Euclidean space equipped with the congruence obtained by translations as different from the same space equipped with the full group of Euclidean isometries. Thurston was lead to a definition of "geometry" by marrying each of these viewpoints. Thurston gave the following definition, which is also discussed and motivated in Scott [37]:

Definition 2.10 (model geometry) An n-dimensional *model geometry* is a pair (X, G), where X is a connected and simply connected n-manifold and G is a Lie group of diffeomorphisms acting transitively on X with compact point stabilisers where

1. G is maximal, that is G is not a proper subgroup of a larger group H acting in the required way on X,
2. There is a discrete subgroup $\Gamma \leq G$ acting freely and properly discontinuously on X as a covering group, such that the quotient $M = X/\Gamma$ is a compact n-manifold.

A manifold M obtained in this way is said to be *modeled on* (X, G), and is called an (X, G) manifold.

Two geometries (X, G) and (X', G') will be considered equivalent if there is a diffeomorphism between X and X' which throws the action of G to that of G'; under these circumstances, G and G' must be isomorphic. If X has the universal covering $\tilde{X}$, then we have the natural geometry $(\tilde{X}, \tilde{G})$, where $\tilde{G}$ consists of the diffeomorphisms of $\tilde{X}$ which are lifts of the elements of G. We thus restrict our attention to manifolds X, where X is simply connected; that is, we pick a representative geometry for locally equivalent geometries having different fundamental groups.

The requirement that G acts transitively on X with compact stabilisers implies there exists a homogeneous Riemannian metric invariant under the action of G on X; this is Lemma 3.4.11 of Thurston [41]. The definition requires that G be maximal so that rather than counting, for example, $(\mathbb{R}^n, \mathbb{R}^n)$ as a separate geometry where $\mathbb{R}^n$ acts on itself by translation, we will instead count $(\mathbb{R}^n, \mathrm{Isom}(\mathbb{R}^n))$ as a geometry.

The second item above can be understood as follows: We can define a projection map $\pi : X \to M$, such that all points in X which are related by the actions of elements in Γ map to the same point in M. The family of maps $\phi : X \to X$ such that $\pi \circ \phi = \pi$, called *covering transformations*, form a group called the *covering group*. With this choice of projection map, it is clear that the covering group is isomorphic to Γ. We have required that Γ act *freely* on X, which means that the identity element of Γ is the only one which stabilises points of X. The requirement that Γ acts *properly discontinuously* means that for every compact subset $C \subseteq X$ the set $\{g \in \Gamma | gC \cap C \neq \emptyset\}$ is finite. Such a group Γ is called a *discrete group of transformations of* X, or simply a *discrete group*. That Γ acts freely and properly discontinuously gives necessary and sufficient conditions for the quotient X/Γ to be a manifold with $X \to X/\Gamma$ a covering projection; see Proposition 3.5.7 of Thurston [41]. If Γ is a discrete subgroup of a Lie group G, then it acts properly discontinuously. Finally, if the quotient X/Γ is compact, then the Γ is called a *cocompact* subgroup.

Before looking at the three-dimensional case, it is instructive to consider the question of classifying the two-dimensional model geometries:

Theorem 2.1 (two-dimensional model geometries) *There are precisely three two-dimensional model geometries: spherical, Euclidean, and hyperbolic.*

PROOF: A version of this proof appears in Thurston [41]. Since X is two-dimensional, $T_p(X)$ is the only tangent two-plane at a point $p \in X$. Thus, the sectional curvature (called the *Gaussian curvature* in the two-dimensional case) is a real-valued function on X. Because X is a homogeneous manifold, the Gaussian curvature must be constant on X. One can show that by multiplying the metric by k, the Gaussian curvature changes by k^2, thus we can scale the metric on X by a constant to obtain one with constant Gaussian curvature 1, 0, or -1. In 1926, H. Hopf showed that the only simply connected, geodesically complete[1] Riemannian n-manifolds with constant sectional curvatures 1, 0, or -1 are the sphere S^n, the flat Euclidean plane E^n, or the hyperbolic space H^n respectively; see for example Section 6.1 of Petersen [33]. These manifolds, together with their isometry groups, give us the three two-dimensional model geometries. □

A similar classification was carried out by Thurston for three-dimensional model geometries, which first appeared in his now famous lecture notes from the late 1970s upon which [41] is based. Scott [37] also discusses the proof, and gives a rather detailed look at geometry in two and three dimensions. The three-dimensional case is approached in a similar fashion to the two-dimensional case, although the details are more difficult. One takes manifolds X equipped with a G-invariant metric, with (X, G) as described for model geometries. We will denote the identity component of the stabiliser subgroup G_x of $x \in X$ as $I(G_x)$. G acts on X by isometries, so the stabiliser G_x acts on $T_x(X)$, preserving the inner product. Thus, G_x is naturally isomorphic to a subgroup of $O(3)$. The connected component $I(G_x)$ is therefore a closed subgroup of SO(3); specifically one of SO(3), SO(2), and the trivial group [41]. The stabiliser G_x is a Lie group of the same dimension (three, one, and zero, respectively). This leads to the statement of Thurston's classification of three-dimensional model geometries.

Theorem 2.2 ((Thurston) three-dimensional model geometries) *There are eight three-dimensional model geometries $(X, Isom(X))$, as follows:*

1. *If the point stabilisers are three-dimensional, X is S^3, E^3, or H^3.*

2. *If the point stabilisers are one-dimensional, X fibres over one of the two-dimensional model geometries, in a way that is invariant under G. There is a G-invariant Riemannian metric on X such that the connection orthogonal to the fibres has curvature 0 or 1.*

   (a) *If the curvature is 0, X is $S^2 \times \mathbb{R}$ or $H^2 \times \mathbb{R}$.*

[1] One says that a manifold is *geodesically complete* if its geodesics exist for all values of their respective affine parameters; that is, all geodesics are infinitely extendible.

(b) *If the curvature is* 1, *we have nilgeometry (which fibres over* E^2*) or the geometry of* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *(which fibres over* H^2*).*

3. *The only geometry with zero-dimensional stabilisers is solvegeometry, which fibres over the line.*

Solutions to five-dimensional general relativity that are foliated by seven of the eight model three-geometries, excepting $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, are presented in Chapter 4. We will explicitly construct compact manifolds modeled on nilgeometry and solvegeometry. In the nilgeometry case the resulting manifold is a circle bundle over the torus; the solvegeometry case is a torus bundle over the circle.

2.4.1 Nilgeometry

For nilgeometry, we take the manifold X to be the Heisenberg group, that is, all 3×3 upper triangular matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$

where $x, y, z \in \mathbb{R}$. The Heisenberg group, denoted Nil, is a nilpotent Lie group. We can identify $(x, y, z) \in \mathbb{R}^3$ with the above matrix, which gives the multiplication $(x, y, z)\cdot(x', y', z') = (x + x', y + y', z + z' + xy')$. We can determine a (left) invariant metric on $\mathbb{R}^3$ under the action of Nil on itself by picking a metric arbitrarily at a point, and using invariance. If we pick $ds^2 = dx^2 + dy^2 + dz^2$ at the origin, the resulting invariant metric is

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2. \tag{2.13}$$

As pointed out in Fagundes [12] and further discussed in Section 2.5, this metric is of Bianchi type II.

The group G of isometries for this geometry contains Nil, but also a group isomorphic to S^1 [37]. If $\rho_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the standard rotation matrix through an angle θ, and we write $\mathbf{x} = (x, y)$, then the metric given in Equation 2.13 is invariant under the action given by

$$(\mathbf{x}, z) \mapsto (\rho_\theta(\mathbf{x}), z + \frac{1}{2}\sin\theta(\cos\theta(y^2 - x^2) - 2xy\sin\theta).$$

An example of a compact three-manifold with geometric structure modeled on Nil can be constructed by taking the quotient of Nil by the subgroup Γ consisting of matrices with only integer entries, following Thurston [41]. If $a, b, c \in \mathbb{Z}$, then

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+a & z+c+ay \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{bmatrix}$$

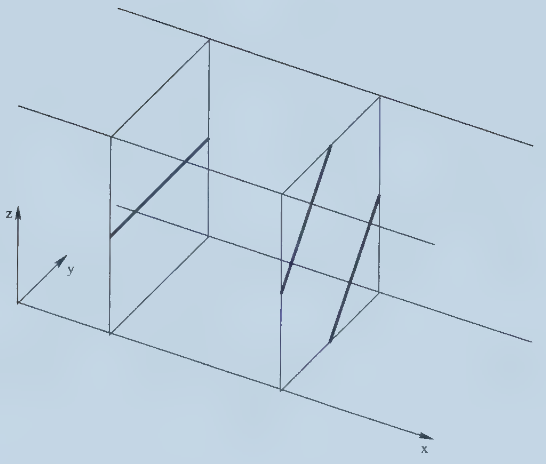

Figure 2.1: Identifications of $\mathbb{R}^3$ for a compact three-manifold modeled on nilgeometry. Based on Figure 3.26 of [41], and also appears as Figure 2 of Cadeau and Woolgar [8].

So two elements (x, y, z) and (x', y', z') from Nil are in the same coset representative of Nil/Γ if $x' - x = a \in \mathbb{Z}$, $y' - y = b \in \mathbb{Z}$, and $z' - z = c + ay$. Consider $x' = x$: at every x, we should identify points in the yz-plane as we would to construct a torus. To identify the tori with $x - x' = a \neq 0$, we take lines $z = k$, k a constant, on the yz-plane at x, and identify them with $z' = k + ay$ at x'. The identifications are shown in Figure 2.1. This manifold is a circle bundle over the torus.

2.4.2 Solvegeometry

The manifold X for solvegeometry is the Lie group Sol, described by the semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$ with the multiplication given by

$$((x, y), z) \cdot ((x', y'), z') = ((x + e^{-z}x', y + e^{z}y'), z + z').$$

A (left) invariant metric on $\mathbb{R}^3$ under the action of Sol on itself is given by $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$, and this metric corresponds to Bianchi type VI$_{-1}$ [12].[2] The identity component of the isometry group for Sol is Sol itself. One can show, following Thurston [41] and Scott [37], that the full group of isometries of Sol has eight connected components. Furthermore, the surfaces of constant z form a two-dimensional foliation of Sol which is preserved by all isometries of Sol.

[2]The choice of parameter for Bianchi VI depends on which enumeration scheme one is using: Fagundes [12] uses the Ellis-MacCallum scheme where this is called Bianchi VI(0). In the Ryan and Shepley scheme [35], this metric corresponds to a parameter value $h = -1$, so we write Bianchi VI$_{-1}$.

To construct a compact three-manifold modeled on Sol, we consider the mapping torus M_ϕ for a diffeomorphism $\phi : T^2 \to T^2$ of the torus to itself. The *mapping torus* M_ϕ is obtained from the "cylinder" $T^2 \times [0, a]$ by identifying the two ends via the map ϕ. The following construction is found in Thurston [41]: Take for example the linear automorphism ϕ of $\mathbb{R}^2$ with matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix is chosen so that it is not the identity, is symmetric with unit determinant and positive trace, and has integer entries. As a result, this matrix has reciprocal eigenvalues, $\frac{1}{2}(3 \pm \sqrt{5})$ in this case, and orthogonal eigenvectors. Finally, such a matrix preserves the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$.

Arrange the universal cover of the torus in $\mathbb{R}^2$ so that the orthogonal eigenspaces of ϕ coincide with the x- and y-axes. The eigenvalues of ϕ are reciprocals, so there is some z_0 such that the transformation

$$\psi(x, y, z) = (e^{-z_0}x, e^{z_0}y, z + z_0)$$

induces the given automorphism ϕ between the copies of $\mathbb{R}^2$, $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{z_0\}$. In performing these identifications, we have taken Γ to be the subgroup of Sol generated by the elements $\{((1,0),0), ((0,1),0), ((0,0),z_0)\}$, that is, unit translations along the x- and y-axes, and the described automorphism of Sol. Figure 2.2 shows these identifications. This manifold is a torus bundle over the circle.

2.5 The nine Bianchi models

Prior to the birth of general relativity, in 1897 Bianchi [5] classified the nine simply transitive three-dimensional Lie groups. Probably the earliest application of these groups in relativity was due to Taub [39]. Subsequently, spacetimes that have spatially homogeneous hypersurfaces became a subject of intense study; see for example Ryan and Shepley [35], or Kramer, Stephani, MacCallum, and Herlt [24]. These spaces are important from a cosmological standpoint because, as previously noted, homogeneous manifolds "look" the same at every point. The assumption of this kind of symmetry is called the *Copernican principle*, which is the basis of much work in cosmology. Fagundes [12] pointed out that the nine Bianchi models are very nearly in correspondence with Thurston's eight model geometries, the key differences being that manifolds with Bianchi type IV symmetries cannot be *compact* three-manifolds. The situation is the same for Bianchi type VI_h when $h \neq -1$.[3] The remaining Bianchi metrics correspond to a model three-geometry; in particular, type II cor-

[3] It is worth noting that the numbering scheme used by Fagundes [12] is different than that of Ryan and Shepley [35]. Fagundes makes a remark that in the Ellis-MacCallum parametrisation scheme, Bianchi type VI(1) reduces to Bianchi type III; this redundancy is not present in Ryan and Shepley's parametrisation scheme, at least formally, because they have explicitly excluded the corresponding type number VI_0.

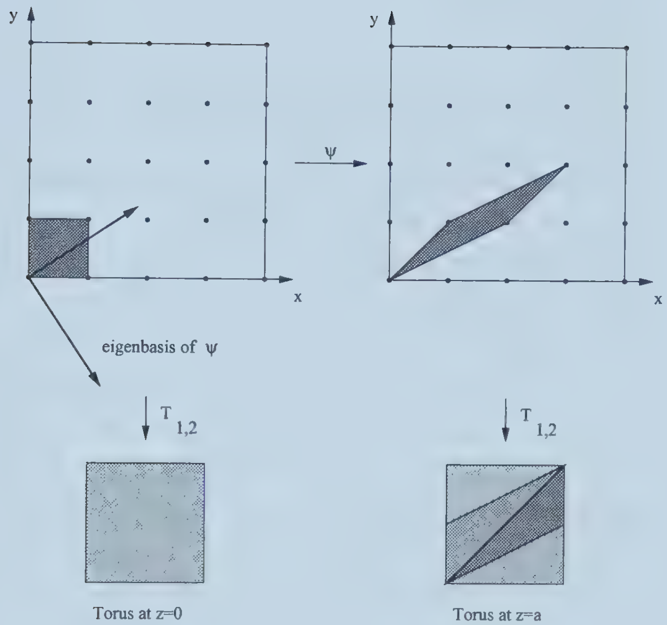

Figure 2.2: Identifications of $\mathbb{R}^3$ for a compact three-manifold modeled on solvegeometry. Appears as Figure 1 of Cadeau and Woolgar [8].

responds to nilgeometry, type VI_{-1} corresponds to solvegeometry, and type VIII corresponds to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

The Bianchi manifolds can be described by a set of Killing vectors $\{\xi_i\}$, an invariant basis $\{X_i\}$ satisfying $[\xi_i, X_j] = 0$, and the invariant one-forms dual to the invariant vector fields $\{\omega^i\}$, where the indices run from 1 to 3. Following the conventions of Ryan and Shepley [35], the Lie algebra has structure constants defined by $[\xi_i, \xi_j] = C^s_{ij}\xi_s$, yielding $[X_i, X_j] = -C^s_{ij}X_s$, and $d\omega^i = \frac{1}{2}C^i_{st}\omega^s \wedge \omega^t$. Table 2.1 gives useful co-ordinate representations for these vectors and one-forms in terms of the co-ordinates $\{x^i\}$, tangent vectors $\partial_i \equiv \partial/\partial x^i$, and dual basis dx^i; this table is adapted from a complete list appearing in [35]. Table 2.1 allows us to construct line elements for three-manifolds modeled on nilgeometry, solvegeometry, and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The line elements for the remaining model geometries are easily constructed: for the $n-$sphere S^n, one starts with the $(n+1)$-dimensional Euclidean metric $ds^2 = dx_1^2 + \ldots + dx_{n+1}^2$ and writes the induced metric for the locus of points satisfying $x_1^2 + \ldots + x_{n+1}^2 = 1$; one can similarly construct the metric for the $n-$dimensional hyperbolic space by starting with $(n+1)$-dimensional Minkowski space $\mathbb{M}^{n+1}$ with the metric $ds^2 = -dx_1^2 + dx_2^2 + \ldots + dx_{n+1}^2$ and writing the induced metric for the locus of points satisfying $-x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = -1$. The line elements for product manifolds is the sum of the line elements of the lower-

dimensional manifolds.

2.6 Thurston's geometrisation conjecture

Loosely, Thurston's geometrisation conjecture states that every compact three-manifold can be split into parts that are modeled on one of the eight model geometries. This splitting proceeds according to two steps: First, the manifold is expressed as a finite connected sum of prime pieces. A three-manifold M is *prime* if any expression of M as the connected sum $M_1 \# M_2$ has one of the factors homeomorphic to S^3. This decomposition is guaranteed due to a result of Kneser [23], and in the orientable case, Milnor [30] showed that the factors in the decomposition are unique. The next step is the Jaco-Shalen-Johannson torus decomposition [21][22], which says that the prime factors above can be cut by a minimal collection of incompressible tori. Thurston's geometrisation conjecture, stated in [40] is that each of the resulting pieces after this decomposition are modeled on one of the eight model geometries. The conjecture has already been shown to be true for a certain class of manifolds known as Haken manifolds [37].

Type II / Nilgeometry	$C^1_{23} = -C^1_{32} = 1$,	$\xi_1 = \partial_2$	
	remaining $C^i_{jk} = 0$.	$\xi_2 = \partial_3$	
		$\xi_3 = \partial_1 + x^3\partial_2$	
	$X_1 = \partial_2$	$\omega^1 = dx^2 - x^1 dx^3$	$d\omega^1 = \omega^2 \wedge \omega^3$
	$X_2 = x^1\partial_2 + \partial_3$	$\omega^2 = dx^3$	$d\omega^2 = 0$
	$X_3 = \partial_1$	$\omega^3 = dx^1$	$d\omega^3 = 0$
Type VI_{-1} / Solvegeometry	$C^1_{13} = -C^1_{31} = 1$	$\xi_1 = \partial_2$	
	$C^2_{23} = -C^2_{32} = -1$	$\xi_2 = \partial_3$	
	remaining $C^i_{jk} = 0$	$\xi_3 = \partial_1 + x^2\partial_2 - x^3\partial_3$	
	$X_1 = e^{x^1}\partial_2$	$\omega^1 = e^{-x^1} dx^2$	$d\omega^1 = \omega^1 \wedge \omega^3$
	$X_2 = e^{-x^1}\partial_3$	$\omega^2 = e^{x^1} dx^3$	$d\omega^2 = -\omega^2 \wedge \omega^3$
	$X_3 = \partial_1$	$\omega^3 = dx^1$	$d\omega^3 = 0$
Type VIII / $\widetilde{\mathrm{SL}}(2,\mathbb{R})$	$C^1_{23} = -C^1_{32} = -1$		
	$C^2_{31} = -C^2_{13} = 1$		
	$C^3_{12} = -C^3_{21} = 1$		
	$\xi_1 = \frac{1}{2}e^{-x^3}\partial_1 + \frac{1}{2}\left[e^{x^3} - (x^2)^2 e^{-x^3}\right]\partial_2 - x^2 e^{-x^3}\partial_3$		
	$\xi_2 = \partial_3$		
	$\xi_3 = \frac{1}{2}e^{-x^3}\partial_1 - \frac{1}{2}\left[e^{x^3} + (x^2)^2 e^{-x^3}\right]\partial_2 - x^2 e^{-x^3}\partial_3$		
	$X_1 = \frac{1}{2}\left[1 + (x^1)^2\right]\partial_1 + \frac{1}{2}\left[1 - 2x^1x^2\right]\partial_2 - x^1\partial_3$		
	$X_2 = -x^1\partial_1 + x^2\partial_2 + \partial_3$		
	$X_3 = \frac{1}{2}\left[1 - (x^1)^2\right]\partial_1 + \frac{1}{2}\left[-1 + 2x^1x^2\right]\partial_2 + x^1\partial_3$		
	$\omega^1 = dx^1 + \left[1 + (x^1)^2\right]dx^2 + \left[x^1 - x^2 - (x^1)^2x^2\right]dx^3$		
	$\omega^2 = 2x^1 dx^2 + \left(1 - 2x^1x^2\right)dx^3$		
	$\omega^3 = dx^1 + \left[-1 + (x^1)^2\right]dx^2 + \left[x^1 + x^2 - (x^1)^2x^2\right]dx^3$		
	$d\omega^1 = -\omega^2 \wedge \omega^3$		
	$d\omega^2 = \omega^3 \wedge \omega^1$		
	$d\omega^3 = \omega^1 \wedge \omega^2$		

Table 2.1: Co-ordinate expressions for relevant Bianchi manifolds, adapted from Table 6.1 of [35]

Chapter 3

General relativity

3.1 Introduction

In 1915, Einstein proposed his theory of general relativity; a mathematical model of space, time, and gravitation. The theory itself has been shown to be experimentally successful, having accounted for such classical results as the precession of Mercury's orbit, and the time delay and deflection of light. More recent tests of the theory include lunar laser ranging, observations of the binary pulsar PSR 1913+16, and currently there is much activity concerning gravitational radiation. Will [44] gives an account of a host of experimental work, and contrasts the predictions made by general relativity with those of competing theories of gravitation, for example the Brans-Dicke scalar field theory. We begin this chapter by stating Einstein's field equations and quickly reviewing how the physical world is described using differential geometric language. We then proceed to discuss black holes, using the Schwarzschild solution as our working model. The chapter concludes with a brief discussion of black hole thermodynamics. The treatment of these subjects draws upon more exhaustive treatments found in texts such as Misner *et al.* [31], Wald [43], Hawking and Ellis [17], and a set of notes prepared by Carroll [10].

General relativity considers time and space to be a (smooth and Hausdorff) four-dimensional semi-Riemannian manifold (M, g), whose metric has Lorentz signature $(-,+,+,+)$. The spacetime manifold satisfies Einstein's equations:

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}, \tag{3.1}$$

where the *Einstein tensor* G_{ab} is defined as the symmetric tensor

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \tag{3.2}$$

The *stress-energy tensor* T_{ab} is a symmetric tensor with vanishing covariant divergence

which represents the energy and momentum densities, and stress associated with all matter and non-gravitational fields.

We will be dealing exclusively with "vacuum solutions" of Equation 3.1, that is, those spacetimes which do not contain energy or matter fields. In this case, $T_{ab} = 0$. The constant Λ is known as the "cosmological constant." The Minkowski spacetime of special relativity is the simplest spacetime; it has the manifold structure of $\mathbb{R}^4$ equipped with the flat metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}dx^\mu dx^\nu, \tag{3.3}$$

and is a vacuum solution to Equation 3.1 when $\Lambda = 0$. For a general spacetime (M, g_{ab}) and any point $p \in M$, one can find a system of co-ordinates so that the metric locally has the form of Equation 3.3 at p.

The Λg_{ab} term was first inserted into Equation 3.1 by Einstein in order that the equations admitted a static solution with the topology of $\mathbb{R} \times S^3$; he later discarded it after Hubble's discovery of the expansion of the universe. For many applications of relativity, for example in astrophysics, one usually takes $\Lambda = 0$, as it can be shown that its effect is negligible on the scale of galaxies and smaller for accepted ranges of Λ. For questions concerning the overall topology of the spacetime manifold, one may choose (as we do) to leave Λ unspecified. There are several reasons to do this.

First, the stress-energy tensor is symmetric and divergence-free, and one can show that the left hand side of Equation 3.1 with Λ unspecified is the most general such tensor which can be locally constructed from the metric and its derivatives up to second order.

A more physical justification for leaving the cosmological term in Equation 3.1 is as follows: consider, for example, the stress-energy tensor for a perfect fluid (no viscosity and no heat conduction) with pressure p, density ρ, and four-velocity u^a, $g_{ab}u^a u^b = -1$:

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}. \tag{3.4}$$

Taking the trace of Equation 3.1, we obtain $-R + 4\Lambda = 8\pi(3p - \rho)$, or

$$-R = 8\pi \left(3(p - \frac{\Lambda}{8\pi}) - (\rho + \frac{\Lambda}{8\pi}) \right). \tag{3.5}$$

One is left with a somewhat arbitrary choice: on the one hand, one could claim that we have a perfect fluid with pressure p and density ρ in the presence of a cosmological constant $\Lambda \neq 0$, or else one could equivalently claim that, in fact, $\Lambda = 0$ and what is present is a perfect fluid with pressure $(p - \frac{\Lambda}{8\pi})$ and density $(\rho + \frac{\Lambda}{8\pi})$.

A third, separate justification arises from quantum field theory, which asserts that the vacuum has a non-zero energy associated with it. One can understand how this can be from a straightforward example in quantum mechanics: in classical physics, a harmonic oscillator

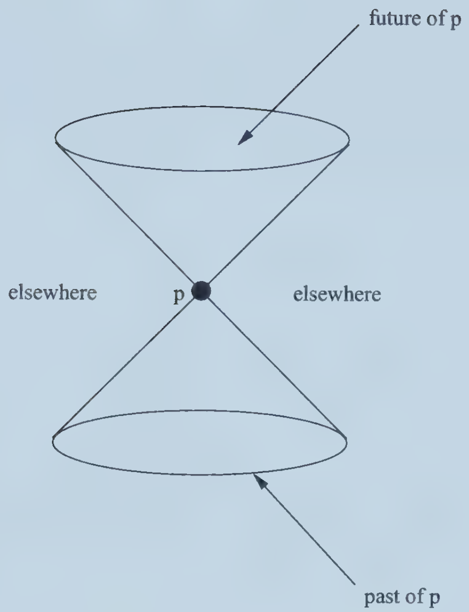

Figure 3.1: The light cone at a point $p \in M$, based on Figure 1.2 of [43]. Points inside the double cone are timelike related to p, points on the null double cone surface are null related to p, and the points outside the double cone are spacelike related to p.

with frequency ω has a minimum energy $E = 0$, which occurs when the oscillator is not in motion. This contrasts with the quantum mechanical analysis of a harmonic oscillator, which asserts that the minimum energy, or ground state, is $E = \hbar\omega/2$. The Λg_{ab} term in Equation 3.1 can be interpreted as a minimum vacuum energy, in the absence of matter or radiation fields.

The metric g_{ab} on the spacetime manifold splits vectors into three classes: a non-zero vector $v^a \in T_p(M)$ is said to be *timelike*, *null*, or *spacelike* if $g_{ab}v^a v^b$ is negative, zero, or positive, respectively. The null vectors in $T_p(M)$ describe a double cone, called a *null* or *light* cone, which separates timelike from spacelike vectors, as depicted in Figure 3.1. Thermodynamically, it is desirable to have a well-defined "arrow of time" at each point in the spacetime; that is, we should be able to continuously classify non-spacelike vectors as either *future-* or *past-directed* at every point in the spacetime. Such a spacetime is called *time orientable*. For our purposes, we restrict our attention to time orientable spacetimes. In general every spacetime has a time orientable double cover [17]. This classification of vectors establishes causal relationships between points in the spacetime, which are treated more formally in Section 3.2.2. Briefly, a signal can be sent between two points of M if

they can be joined by a non-spacelike curve, that is, a piecewise differentiable curve whose tangent vector is nowhere spacelike. Consider an arbitrarily parametrised, differentiable, timelike curve $\gamma(t)$ with associated tangent vectors v^a, and with increasing parameter value representing motion in the future direction. To compute the *four-velocity*, we reparametrise the curve according to the *proper time* τ,

$$\tau = \int \sqrt{-g_{ab}v^a v^b}\,dt, \tag{3.6}$$

which corresponds to the reading of a clock that an observer travelling along that curve carries. The tangent vector u^a at points along this reparametrised curve is called its *four-velocity*, and from this reparametrisation, it satisfies $g_{ab}u^a u^b = -1$.

When only gravitational forces are present, general relativity dictates that the path of a particle through spacetime, called its *world line*, must be a geodesic. World lines for massive particles must be timelike, while photons follow null paths. Geodesics are those curves $x^a(\alpha)$ which parallel transport their own tangent vectors $u^a = \frac{dx^a}{d\alpha}$, so that

$$u^a \nabla_a u^b \propto u^b. \tag{3.7}$$

One can then introduce a new parameter $\lambda = \lambda(\alpha)$ to obtain the *geodesic equation*

$$u^a \nabla_a u^b = 0; \tag{3.8}$$

such a parameter is called an *affine parameter.* Equation 3.8 is written in a co-ordinate basis as

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right) = 0. \tag{3.9}$$

These are differential equations for the co-ordinate trajectories of the freely falling particle, parametrised by the affine parameter λ. Along timelike geodesics, proper time is an affine parameter.

Suppose a spacetime possesses a symmetry described by a Killing vector field ξ^a satisfying Killing's equation 2.12 from Section 2.3:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

From this symmetry, one can show that the quantity $\xi_a u^a$ is conserved for a particle moving on a geodesic with tangent u^a, as

$$u^b \nabla_b (\xi_a u^a) = u^b u^a \nabla_b \xi_a + \xi_a u^b \nabla_b u^a = 0, \tag{3.10}$$

where the first term vanishes as a consequence of Killing's equation 2.12:

$$\begin{aligned} 0 &= u^a u^b \left(\nabla_a \xi_b + \nabla_b \xi_a\right) \\ &= 2u^a u^b \nabla_b \xi_a. \end{aligned}$$

The second term in 3.10 vanishes directly as a result of Equation 3.8. In particular, suppose the components of the metric tensor in some co-ordinate basis were independent of a given co-ordinate, say $y = x^K$, for some K. By following the discussion at Equation 2.12, we immediately have that the vector $\xi = \partial/\partial y$ is the relevant Killing vector, with components $\xi^\mu = \delta^\mu{}_K$. Then a particle in free fall has $\xi_\mu u^\mu = \xi^\mu u_\mu = \delta^\mu{}_K u_\mu = u_K$ as a constant of its motion. Equivalently, we could write this in terms of the four-momentum $p^a \equiv \mu u^a$, and write that the quantity p_K is conserved for a particle in free fall. As a result, it is simpler to analyse the motion of particles in spacetimes possessing symmetries, as will be demonstrated below.

3.2 Black holes

Laplace foresaw the existence of black holes when, in 1795, he wrote *Le Système du monde* [26] in which he recognised that the trapping of light by sufficiently massive objects was a consequence of Newton's theories of gravity and light. In 1916, Schwarzschild wrote down the metric describing the field outside a static, uncharged, spherically symmetric star of mass M,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.11}$$

For a non-rotating and uncharged star of radius $a > 2M$, this metric describes the field outside the star with $r \geq a$. This spacetime is a vacuum solution to Einstein's equations 3.1 with $\Lambda = 0$, and Birkhoff's theorem says that any vacuum spacetime that is spherically symmetric is locally equivalent to a piece of this geometry. Sufficiently large stars that have exhausted their nuclear fuel will collapse through the gravitational radius $r = 2M$ until the entire mass of the star is concentrated within an arbitrarily small radius, and a black hole is formed: a region in the spacetime from which neither light nor massive bodies can escape. We will justify this statement in this section, and along the way use the Schwarzschild solution as a model to exhibit some of the properties of black holes.

3.2.1 The Schwarzschild solution

Looking at Equation 3.11, one immediately sees that the metric becomes singular at $r = 0$, where g_{tt} becomes infinite, and also at $r = 2M$, where g_{rr} become infinite. One has to ask if these singularities in the metric arise from a poor choice in the co-ordinate system, or whether there is indeed some underlying co-ordinate independent pathology in the manifold that the co-ordinates cover. A point in the manifold described by a system of co-ordinates might be said to be "singular" if the curvature of the manifold was infinite there. Curvature

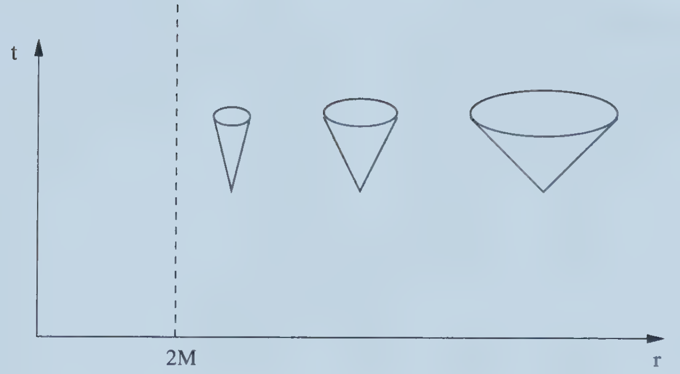

Figure 3.2: Null cones in the (t, r) plane in Schwarzschild co-ordinates. Based on a figure in [10].

is measured by the Riemann tensor R_{abcd}, and its components depend on the system of co-ordinates chosen; however, one could form curvature scalars that depend on the Riemann tensor, for example $R = R^a{}_a$, $R^{ab}R_{ab}$, $R^{abcd}R_{abcd}$, $R_{abcd}R^{cdef}R_{ef}{}^{ab}$, and so on, and these quantities would be independent of the co-ordinate system chosen. If these curvature scalars diverged, one would be certain that at least one component of the Riemann tensor would be infinite, independent of any co-ordinate system it was calculated in. An often used curvature invariant is the *Kretschmann scalar*, which is defined as $I = R^{abcd}R_{abcd}$. One finds that $I = 12M^2/r^6$ for the Schwarzschild geometry, which means that the underlying spacetime is indeed pathological at $r = 0$. We say that the divergence of curvature scalars is a sufficient condition for establishing the presence of a true singularity; at these points, the theory breaks down and one cannot do meaningful physical calculations.

It is a more difficult chore to establish that the points at $r = 2M$ are not singular. It will be demonstrated that $r = 2M$ is only a co-ordinate singularity, although this surface does have interesting properties. First, consider the radial null geodesics (θ and ϕ are constant); they satisfy

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}}, \tag{3.12}$$

which leads to

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}. \tag{3.13}$$

From this equation, we can obtain a picture of the light cones in the $t - r$ plane, shown in Figure 3.2. As $r \to 2M$, $dt/dr \to \pm\infty$, and the light cones appear to close. One might intuit from this that it is not possible to reach the event horizon in a finite amount of time, owing to the "closing" of the light cones. This is a half-truth, for we will now show that it

takes an infinite lapse of *co-ordinate* time for a test particle to reach the event horizon, but a finite lapse of the test particle's *proper* time. Qualitatively, this means that if one is at $r > 2M$ observing a particle falling toward the event horizon that the particle would never be seen to reach it. However, the particle itself would in fact reach, and cross, the event horizon in a finite amount of its proper time. Interestingly, before the term "black hole" was coined, the term "frozen star" was used as a result of this effect.

Now consider the motion of a test particle in the spacetime described by the Schwarzschild metric 3.11. Notice that under the transformation $\theta \mapsto \pi - \theta$ that the metric is unchanged. Therefore, if the initial position and tangent vector of a geodesic lie in the "equatorial plane" $\theta = \pi/2$, then the entire geodesic must lie in this plane. Departing from the equatorial plane would introduce a preferred direction, which is contrary to the symmetry of the spacetime. As we can always rotate the co-ordinates of the S^2 part of the spacetime to be such that a particle's initial position and four-velocity lie in this equatorial plane, without loss of generality we can consider only those geodesics with $\theta = \pi/2$ and $u^\theta = d\theta/d\lambda = 0$. Let us consider a massive particle moving on timelike geodesics, so that (for $\theta = \pi/2$)

$$\begin{aligned} g_{\alpha\beta}u^\alpha u^\beta &= -1 \\ -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 &= -1. \end{aligned} \tag{3.14}$$

We know from the t and ϕ independence of the metric that $E \equiv -u_t$ and $L \equiv u_\phi$ are constants of the motion, so that we have

$$E = -u_t = -g_{\alpha t}u^\alpha = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}, \tag{3.15}$$

and

$$L = r^2\frac{d\phi}{d\lambda}. \tag{3.16}$$

Equation 3.14 then becomes

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right). \tag{3.17}$$

Equations 3.15, 3.16, and 3.17 are general equations useful for analysing orbits of massive particles in this geometry. Let us specialise to the case of a particle initially at rest, falling in radially from infinity. The assumption that the particle is initially at rest gives that $E = 0$, and the radial infall assumption gives that $L = 0$. To obtain the proper time, it is most convenient to put the parameter $\lambda = 0$ at $r = 0$, and we have from Equation 3.17 that

$$d\lambda = -\sqrt{\frac{r}{2M}}dr, \tag{3.18}$$

where the minus sign is chosen so that a negative change in r results in a positive change in λ (infall). Then,

$$\frac{\lambda(r)}{2M} = \frac{1}{2M}\int_0^{\lambda(r)} d\lambda' = -\frac{1}{2M}\int_0^r \sqrt{\frac{r'}{2M}}dr' = -\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2}. \tag{3.19}$$

This gives us the proper time as a function of r. Now for t,

$$\frac{dt}{dr} = \left(\frac{dt}{d\lambda}\right)\left(\frac{d\lambda}{dr}\right), \tag{3.20}$$

so that

$$\begin{aligned} t = \int dt &= -\int \sqrt{\frac{r}{2M}}\frac{1}{1-2M/r}dr \\ &= -\frac{1}{\sqrt{2M}}\int \frac{r^{3/2}}{r-2M}dr \end{aligned} \tag{3.21}$$

which is easily evaluated by partial fractions after setting $u = \sqrt{r}$. The resulting integral is

$$\frac{t(r)}{2M} = -\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2} - 2\left(\frac{r}{2M}\right)^{1/2} + \ln\frac{\sqrt{r/2M}+1}{\sqrt{r/2M}-1}. \tag{3.22}$$

Notice that the expression for co-ordinate time becomes infinite as $r \to 2M$, but there is no such problem with the proper time, the time that would be measured on a clock carried by the infalling test particle. An observer remaining stationary far away from the black hole has their proper time approximated by the co-ordinate time; thus, an outside observer would not witness the particle reaching $r = 2M$. Instead, they would see the test particle asymptotically approach event horizon without reaching it. However, the test particle's clock ticks off its proper time; the particle experiences passing through $r = 2M$ with no immediate ill effects. This is a temporary situation, however, as now the test particle is unavoidably drawn towards the singularity, experiencing extreme tidal forces along the way.

These calculations suggest that there is not anything wrong with the spacetime manifold at $r = 2M$, but rather that the co-ordinates are bad there. Moreover, they suggest that the affine parameter of infalling geodesics might be useful in producing a well-behaved set of co-ordinates. Returning to Equation 3.13, one finds that

$$t = \pm r^* + \text{constant}, \tag{3.23}$$

where the *tortoise co-ordinate* r^* is defined as

$$r^* = r + 2M\ln\left(\frac{r}{2M} - 1\right), \tag{3.24}$$

with the restriction that $r > 2M$. Using these co-ordinates,

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2, \tag{3.25}$$

where r is implicitly given by r^* via Equation 3.24. Next, we introduce the *null co-ordinates*

$$u = t + r^* \tag{3.26}$$
$$v = t - r^*, \tag{3.27}$$

so that the ingoing null geodesics are $u =$ constant, and the outgoing ones are $v =$ constant. The (r, u) co-ordinate system is called *Eddington-Finkelstein co-ordinates*, and in this system the metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 + 2dudr + r^2 d\Omega^2. \tag{3.28}$$

In these co-ordinates, the conditions for radial null curves analogous to Equations 3.12 and 3.13 are

$$\frac{du}{dr} = 0, \tag{3.29}$$

when we are considering ingoing null geodesics, and

$$\frac{du}{dr} = 2\left(1 - \frac{2M}{r}\right)^{-1} \tag{3.30}$$

when we consider the outgoing null geodesics. The null cones in the (r, u) plane are shown in Figure 3.3. One can see in these co-ordinates that at $r < 2M$, all future-directed paths must be in the direction of decreasing r. One could have intuited this by looking at the original metric in Equation 3.11; for $r < 2M$, the radial co-ordinate changes character from spacelike to timelike, as g_{rr} is negative there. One cannot avoid moving in the direction of decreasing r inside the black hole any more than one can avoid moving in the direction of increasing t outside it. The surface $r = 2M$ is known as the *future event horizon*; it is the boundary of those events which can be observed at points with $r > 2M$.

So far we have only shown that certain future directed curves can cross the event horizon. We could alternatively have picked v in place of u in the discussion leading up to Equation 3.28, and shown an analogous fact for past-directed curves. The *Kruskal-Szekeres* co-ordinates make use of both the ingoing and outgoing null co-ordinates u (Equation 3.26) and v (Equation 3.27). Writing the original metric in terms of these co-ordinates, one obtains

$$ds^2 = \left(1 - \frac{2M}{r}\right) dudv + r^2 d\Omega^2. \tag{3.31}$$

Here, r is defined implicitly by u and v according to

$$\frac{1}{2}(u - v) = r + 2M \ln\left(\frac{r}{2M} - 1\right), \tag{3.32}$$

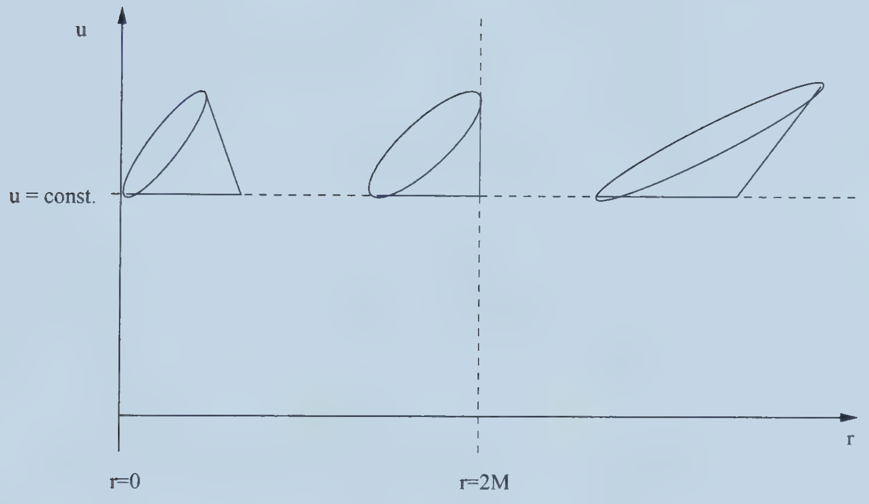

Figure 3.3: Null cones in Eddington-Finkelstein co-ordinates. Based on a a figure from [10].

and $r = 2M$ has been pushed out to $u = -\infty$ or $v = \infty$, which is not useful for understanding this surface. However, if we change co-ordinates again via

$$u' = e^{u/4M} \tag{3.33}$$
$$v' = e^{-v/4M}, \tag{3.34}$$

where in terms of the original co-ordinates,

$$u' = \sqrt{\frac{r}{2M} - 1}\, e^{(r+t)/4M} \tag{3.35}$$
$$v' = \sqrt{\frac{r}{2M} - 1}\, e^{(r-t)/4M}, \tag{3.36}$$

one obtains the metric

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du' dv' + r^2 d\Omega^2. \tag{3.37}$$

Making the further transformation

$$r' = \frac{1}{2}(u' - v') \tag{3.38}$$
$$= \sqrt{\frac{r}{2M} - 1}\, e^{r/4M} \cosh(t/4M), \tag{3.39}$$

and

$$t' = \frac{1}{2}(u' + v') \tag{3.40}$$
$$= \sqrt{\frac{r}{2M} - 1} e^{r/4M} \sinh(t/4M), \tag{3.41}$$

one obtains the final metric in *Kruskal-Szekeres co-ordinates*,

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dt'^2 + dr'^2) + r^2 d\Omega^2. \tag{3.42}$$

The original r co-ordinate is defined via the equation

$$(r'^2 - t'^2) = \left(\frac{r}{2M} - 1\right) e^{r/2M}. \tag{3.43}$$

In these co-ordinates, the radial null curves satisfy $t' = \pm r' +$ constant, and the $r = 2M$ surface corresponds to $t' = \pm r'$. Surfaces of constant r are the hyperbolae $r'^2 - t'^2 =$ constant, and surfaces of constant t are the lines $t'/r' = \tanh(t/4M)$. Initially, our co-ordinates for the Schwarzschild spacetime were valid only where $r > 2M$. If we allow the new co-ordinates to range over all values except where $r = 0$, that is, $-\infty < r' < \infty$ and $t'^2 < r'^2 + 1$, we obtain the maximal extension of the Schwarzschild spacetime. A diagram of this spacetime is shown in Figure 3.4. Every point on this diagram represents a two-sphere.

Region I corresponds to the exterior Schwarzschild spacetime, where the analysis began, and region II represents the interior of the black hole. Regions III and IV are new; region III represents a region of the spacetime from which things can escape but not enter, sometimes called a "white hole." Region IV is a region similar to I, however, no timelike traveler can travel between regions I and IV. The boundary of region II is called a *future event horizon*, and the boundary of region III is called a *past event horizon*. This spacetime is maximally extended; no other regions of the Schwarzschild spacetime can be found.

3.2.2 Conformal infinity, and Hawking's black hole definition

In the previous section, we made a demonstration of the idea that a "black hole" is a region in a spacetime from which no future-directed null or timelike curve can escape, using the Schwarzschild solution. Owing to work that was started in the mid-1960s by Penrose, and in the early 1970s by Hawking, one can rigorously define the notion of a black hole in the case of asymptotically flat spacetimes. This work post-dated the initial work concerning black holes, and as will be seen, the proposed definition does not include all black holes. In this section, we shall present a brief overview of this formalism; Section 3.2.3 will discuss a case where the formalism doesn't apply, even though the spacetime should clearly be interpreted as a black hole.

We begin by defining the notion of two *conformal* spacetimes. Two spacetimes $(\bar{M}, \bar{g}_{ab})$ and (M, g_{ab}) are said to be *conformal* if there is a C^∞ diffeomorphism $\psi : M \to \bar{M}$ such that $\psi_* g_{ab} = \bar{g}_{ab} = \Omega^2 g_{ab}$ on the image of M, for some non-zero, sufficiently differentiable function Ω. The structure of the null cone field is preserved between two conformal spacetimes, but the curvature tensors that we have employed are not conformally invariant.[1] In

[1] One *can*, however, define a curvature tensor which is conformally invariant, known as the *Weyl tensor*.

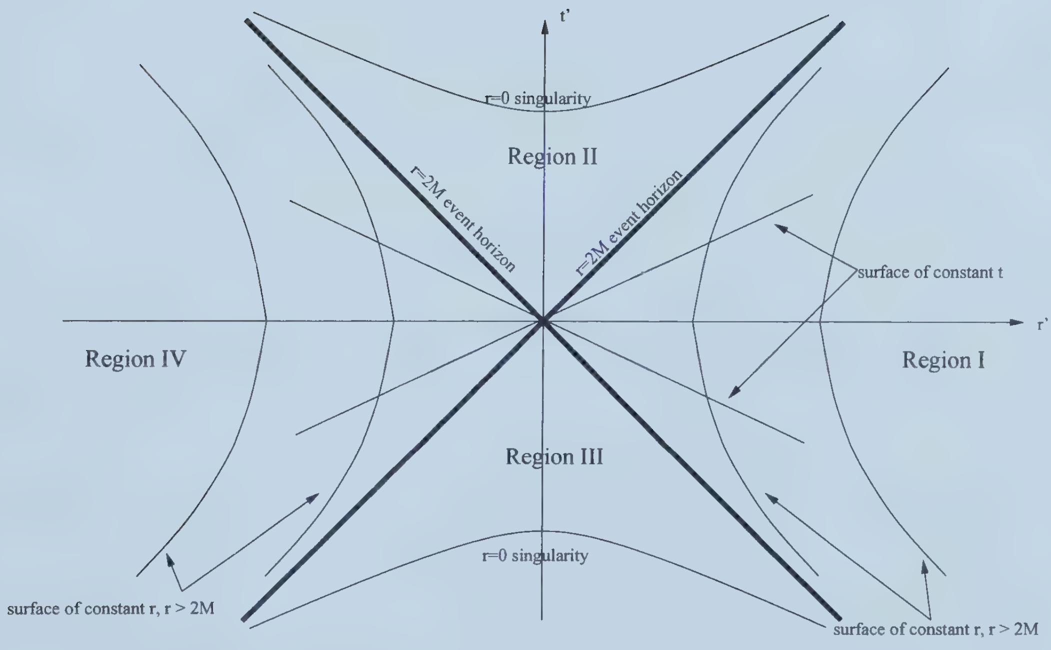

Figure 3.4: Kruskal diagram for the maximally extended Schwarzschild spacetime, based on Figure 24(i) of [17]

order to study the properties "at infinity" of a spacetime, Penrose introduced the idea of extending the spacetime manifold to include points "at infinity" by employing a new system of co-ordinates in which one can interpret points with finite co-ordinate values as being "at infinity" in the original spacetime. To illustrate, we begin by considering Minkowski space, with the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{3.44}$$

where $d\Omega^2$ represents the two-sphere metric. We begin by employing the null co-ordinates defined by $u = t + r$ and $v = t - r$, with $u \geq v$ and $u, v \in \mathbb{R}$. The metric becomes

$$ds^2 = -dudv + \frac{1}{4}(u - v)^2 d\Omega^2. \tag{3.45}$$

Now we make a co-ordinate change so that as u, v approach infinity, the new co-ordinates take finite values. We take the system

$$\begin{aligned} U &= \arctan u \\ V &= \arctan v, \end{aligned} \tag{3.46}$$

where we now have that the points in the original spacetime correspond to values of U, V obeying $-\pi/2 < U, V < \pi/2$, with $V \leq U$. The metric in this system of co-ordinates becomes

$$ds^2 = \sec^2 U \sec^2 V \left(-dUdV + \frac{1}{4}\sin^2(U - V)d\Omega^2\right). \tag{3.47}$$

This physical metric is conformal to the unphysical metric given by the line element

$$d\bar{s}^2 = -4dUdV + \sin^2(U - V)d\Omega^2, \tag{3.48}$$

and we can introduce co-ordinates on the unphysical metric by $t' = U + V$ and $r' = U - V$, where the ranges of the new co-ordinates are given by $-\pi < t' + r' < \pi$, $-\pi < t' - r' < \pi$, and $r' \geq 0$. In these new co-ordinates, the unphysical metric is given by

$$d\bar{s}^2 = -dt'^2 + dr'^2 + \sin^2 r' d\Omega^2. \tag{3.49}$$

It is interesting that the resulting unphysical metric is a portion of the metric on $\mathbb{R} \times S^3$. While it is not a vacuum solution (as the original metric was), it *is* a solution to the field equations 3.1 with a perfect fluid and a cosmological constant, called the Einstein static universe (ESU).

The boundary of this conformal manifold consists of the lines in the (U, V) plane $U = \pi/2$, $V = -\pi/2$, and the points $U = V = \pi/2$ and $U = V = -\pi/2$. Figure 3.5 shows a diagram of the conformal metric, known as the *Penrose diagram*, suppressing the

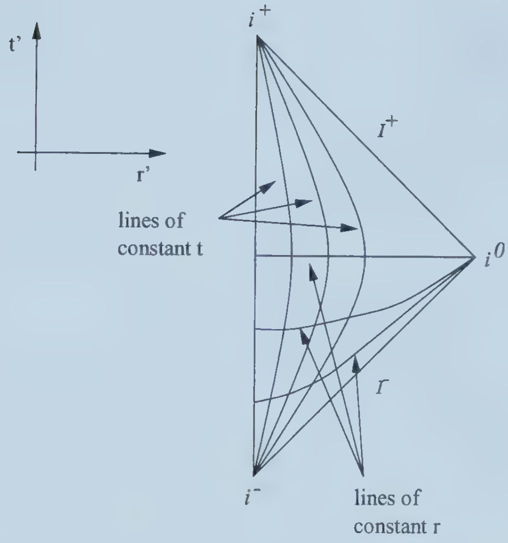

Figure 3.5: Penrose diagram for Minkowski space, based on Figure 15(ii) of [17]

Region	Symbol	(U, V)	(t', r')
Future timelike infinity	i^+	$U = \pi/2,\ V = \pi/2$	$t' = \pi$
Past timelike infinity	i^-	$U = -\pi/2,\ V = -\pi/2$	$t' = -\pi$
Future null infinity	$\mathscr{I}^+$	$U = \pi/2$	$t' = \pi - r',\ 0 < r' < \pi$
Past null infinity	$\mathscr{I}^-$	$V = -\pi/2$	$t' = -\pi + r',\ 0 < r' < \pi$
Spacelike infinity	i^0	$U = \pi/2,\ V = -\pi/2$	$t' = 0,\ r' = \pi$

Table 3.1: Regions of infinity for Minkowski space

two-spheres at every point. Any future-directed timelike geodesic in Minkowski space approaches i^+ (respectively i^-) for indefinitely large positive (negative) values of its affine parameter, and one refers to i^+ as future timelike infinity, and i^- as past timelike infinity. Similarly, null geodesics begin at past null infinity, $\mathscr{I}^-$, and end at future null infinity $\mathscr{I}^+$. Spacelike geodesics begin and end at spacelike infinity i^0. Together these sets represent the boundary of the conformal metric, and are referred to as "conformal infinity." Table 3.1 below summarises the regions of infinity in each of the co-ordinate systems that have been employed.

One can follow the same steps used to obtain the Penrose diagram for Minkowski space, to obtain the Penrose diagram for the maximally-extended Schwarzschild spacetime, shown in Figure 3.6. Beginning with the metric of Equation 3.42, one can employ the co-ordinates

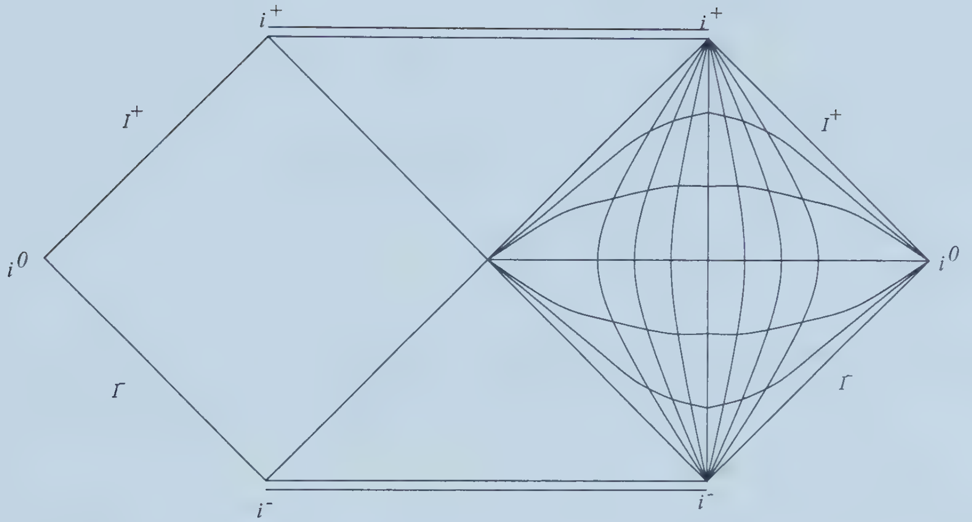

Figure 3.6: The Penrose diagram for the maximally extended Schwarzschild spacetime, based on Figure 34.3 of [31].

(ψ, ξ) in favour of (t', r'), defined as follows:

$$\begin{aligned} t' + r' &= \tan \frac{1}{2}(\psi + \xi) \\ t' - r' &= \tan \frac{1}{2}(\psi - \xi). \end{aligned} \tag{3.50}$$

One then obtains

$$(1 - r/2M)e^{r/2M} = t'^2 - r'^2 = \tan \frac{1}{2}(\psi + \xi) \tan \frac{1}{2}(\psi - \xi), \tag{3.51}$$

and then the metric

$$ds^2 = \frac{32M^3}{r} \frac{e^{-r/2M}(-d\psi^2 + d\xi^2)}{4\cos^2 \frac{1}{2}(\psi + \xi) \cos^2 \frac{1}{2}(\psi - \xi)} + r^2 d\Omega^2. \tag{3.52}$$

One can make a few interesting observations from the Penrose diagram in Figure 3.6. First, the structure of conformal infinity is essentially the same as that of Minkowski space (except for differences in the differentiable structure at i^0, discussed by Ashtekhar and Hansen in [2]). Further, the region within the future event horizon is invisible from $\mathscr{I}^+$. It is this property which leads to a definition of a black hole in an asymptotically flat spacetime.

In order to formulate this definition, one needs definitions for a variety of causal relationships between events and regions of a spacetime. We say that for points a and b in a spacetime that $a \ll b$, or *a chronologically precedes* b, if there exists at least one smooth,

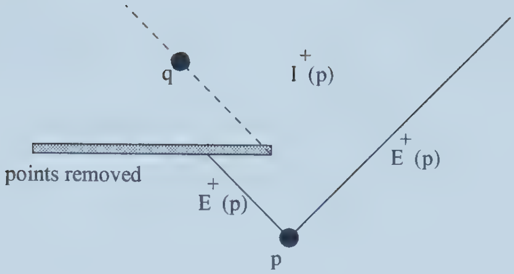

Figure 3.7: Here we have removed points from Minkowski space, and as a result not all points forming the boundary of $I+(p)$ and $J^+(p)$ are in $E^+(p)$. For example, $q \in \partial I^+(p)$ but $q \notin E^+(p)$. Based on Figure 34 of [17].

future-directed timelike curve that extends from a to b. A *causal curve* $\gamma(\lambda)$ is any smooth curve whose tangent vector is nowhere spacelike. We say $a \prec b$, or a *causally precedes* b, if there is at least one future-directed causal curve that extends from a to b. $J^-(a) \equiv \{c|c \prec a\}$ is called the *causal past of* a; the causal future $J^+(a)$ is defined similarly. If S is some subset of the spacetime, then $J^-(S) \equiv \{c|c \prec a \text{ for at least one } a \in S\}$ is the causal past of S, and similarly for the causal future. Analogous statements can be made for the *chronological future* I^+ and *chronological past* I^-, where these are sets defined using the "chronologically precede" notion, rather than "causally precede." A set $\mathscr{S}$ is said to be *acausal* if $J^+(\mathscr{S}) \cap \mathscr{S} = \emptyset$, that is, no two points of $\mathscr{S}$ have timelike or null separation; similarly, an *achronal* set is one in which no two points have timelike separation. Finally, the *future horismos of* $\mathscr{S}$, written $E^+(\mathscr{S})$, is the set $E^+(\mathscr{S}) \equiv J^+(\mathscr{S}) \setminus I^+(\mathscr{S})$. If $p \in M$, $E^+(p)$ lies in the boundary of $J^+(p)$ and $I^+(p)$. In Minkowski space $E^+(p)$ is the surface of the future null cone at p, $J^+(p)$ consists of the points lying within or on the surface of the future null cone, and $I^+(p)$ consists of those points strictly within the future null cone. This example is deceptive, because in more complicated spacetimes $E^+(p)$ is not necessarily the boundary of $J^+(p)$ and $I^+(p)$; see Figure 3.7.

A time- and space-orientable space (M, g_{ab}) is called *asymptotically simple* if:

1. If there is an imbedding $\psi : M \to \bar{M}$ which imbeds M as a manifold with smooth boundary[2] ∂M in the (strongly causal[3]) spacetime $(\bar{M}, \bar{g}_{ab})$, and,

2. (M, g_{ab}) is conformally related to the spacetime $(\bar{M}, \bar{g}_{ab})$, such that $\partial M \subseteq \bar{M} \setminus M$, with

[2]From a topological standpoint, it is perhaps rather confusing to speak of ∂M, as the spacetime manifold M does not necessarily have a boundary. The symbol ∂M refers to the boundary at infinity of the manifold *after* conformal completion, which are points in $\bar{M}$.

[3]The *strong causality* condition on a spacetime (M, g_{ab}) states that there is a neighbourhood of every point $p \in M$ which no non-spacelike curve will intersect more than once.

a sufficiently smooth conformal factor Ω for the metric $d\bar{s}^2 = \Omega^2 ds^2$ which satisfies $\Omega > 0$ on M, $\Omega = 0$ on ∂M, and $\bar{\nabla}_a \Omega \neq 0$, and,

3. Every null geodesic in M has two endpoints on ∂M.

Furthermore, (M, g_{ab}) is called *asymptotically simple and empty* if it also satisfies the condition that $R_{ab} = 0$ in a neighbourhood of ∂M in M. As we have seen, Minkowski space with its conformal boundary is asymptotically simple and empty. The Schwarzschild spacetime, however, is not; null geodesics which hit the singularity do not end on $\mathscr{I}^+$. We have seen that the Schwarzschild spacetime is asymptotically Minkowski, and we say that a space (M, g_{ab}) is *weakly asymptotically simple and empty* if there is an asymptotically simple and empty space (M', g'_{ab}) and a neighbourhood U' of $\partial M'$ in M' such that $U' \cap M'$ is isometric to an open set U of M. One can show that spacetimes that possess this property are asymptotically Minkowski, or *asymptotically flat.*

A point p is called a *future endpoint* of a future-directed non-spacelike curve $\gamma : F \to M$, with F a connected subset of $\mathbb{R}$, if for every neighbourhood U of p there is a $t \in F$ such that $\gamma(t_1) \in U$ for every $t_1 \in F$ with $t_1 \geq t$. A non-spacelike curve is *future-inextendible* (respectively, *future-inextendible in a set* $\mathscr{S}$) if it has no future endpoint (no future endpoint in $\mathscr{S}$). The notion of *past-inextendible* is similarly defined. The *future Cauchy development* or *domain of dependence* of a set $\mathscr{S}$, $D^+(\mathscr{S})$, is the set of all points $p \in M$ such that every past-inextendible non-spacelike curve through p intersects $\mathscr{S}$.

The *edge* of an achronal set $\mathscr{S}$ is defined as those points y in the closure of $\mathscr{S}$, $y \in \bar{\mathscr{S}}$, such that in every neighbourhood U of y, there are points $x \in I^-(y) \cap U$ and $z \in I^+(y) \cap U$ which can be joined by a timelike curve in U which does not intersect $\mathscr{S}$. An acausal set $\mathscr{S}$ with no edge is called a *partial Cauchy surface*; this is a spacelike hypersurface which no non-spacelike curve intersects more than once.

An achronal set $\mathscr{S}$ is called *future-trapped* (respectively, *past-trapped*) if $E^+(\mathscr{S})$ $(E^-(\mathscr{S}))$ is compact. In n dimensions, a *closed trapped surface* $\mathscr{T}$ is a spacelike $(n-2)$-dimensional surface which has the property that the two families of null geodesics orthogonal to $\mathscr{T}$ are converging at $\mathscr{T}$. In four dimensions, given a normalised set of vectors at $p \in \mathscr{T}$, $\{N_1^a, N_2^a, Y_1^a, Y_2^b\}$, where N_i^a are null vectors, and Y_i^a are orthogonal spacelike vectors, one defines the two null second fundamental forms of $\mathscr{T}$ as

$$_n\chi_{ab} = -N_{nc;d}(Y_1^c Y_{1a} + Y_2^c Y_{2a})(Y_1^d Y_{1b} + Y_2^d Y_{2b}), \tag{3.53}$$

where n is either 1 or 2. The two families of null geodesics are converging at $\mathscr{T}$ if $_n\chi_{ab} g^{ab} < 0$ for both $n = 1$ and $n = 2$.

The presence of singularities which are not hidden by an event horizon leads to difficulties solving the initial value problem for Einstein's field equations, and thus being able to

calculate the future time-evolution of a system. Suppose that we have a spacetime (M, g_{ab}) which is weakly asymptotically simple and empty, along with the conformal space $(\tilde{M}, \tilde{g}_{ab})$, into which M is imbedded as a manifold-with-boundary $\bar{M} = M \cup \partial M$, where the boundary ∂M in $\tilde{M}$ consists of the null surfaces $\mathscr{I}^+$ and $\mathscr{I}^-$. Let $\mathscr{S}$ be a partial Cauchy surface for the spacetime (M, g_{ab}). (M, g_{ab}) is said to be *future asymptotically predictable from* $\mathscr{S}$ if $\mathscr{I}^+$ is contained in the closure of $D^+(\mathscr{S})$ in the conformal manifold $\tilde{M}$. Both Minkowski space and the Schwarzschild spacetime are future asymptotically predictable. One can show (for example, Proposition 9.2.1 of [17]) that if (M, g_{ab}) is future asymptotically predictable from a partial Cauchy surface $\mathscr{S}$, and the spacetime satisfies the energy condition[4] $R_{ab}K^aK^b \geq 0$ for all null vectors K^a, that a closed trapped surface $\mathscr{T}$ in $D^+(\mathscr{S})$ cannot intersect $J^-(\mathscr{I}^+)$. Then, in a spacetime satisfying these conditions, a closed trapped surface in $D^+(\mathscr{S})$ must therefore be contained in $M \setminus J^-(\mathscr{I}^+)$, and the boundary of the region from which particles or photons can escape to infinity in the future direction is called the *future event horizon*, $\partial J^-(\mathscr{I}^+)$.

The definition of asymptotic predictability is strengthened to guarantee that this property is not destroyed under perturbations of points in the neighbourhood of the event horizon. A spacetime is *strongly future asymptotically predictable* from a partial Cauchy surface $\mathscr{S}$ if it also has the property $J^+(\mathscr{S}) \cap \bar{J}^-(\mathscr{I}^+) \subseteq D^+(\mathscr{S})$. Proposition 9.2.3 of [17] states that for such a spacetime, there is a homeomorphism

$$\alpha : (0, \infty) \times \mathscr{S} \to D^+(\mathscr{S}) \setminus \mathscr{S}, \tag{3.54}$$

such that for each $\tau \in (0, \infty)$, $\mathscr{S}(\tau) \equiv (\{\tau\} \times \mathscr{S})$ is a partial Cauchy surface such that:

1. for $\tau_2 > \tau_1$, $\mathscr{S}(\tau_2) \subset I^+(\mathscr{S}(\tau_1))$, and,

2. for each τ, the edge of $\mathscr{S}(\tau)$ in the conformal manifold $\tilde{M}$ is a spacelike two-sphere $\mathscr{Q}(\tau)$ in $\mathscr{I}^+$ such that for $\tau_2 > \tau_1$, $\mathscr{Q}(\tau_2)$ is strictly to the future of $\mathscr{Q}(\tau_1)$, and,

3. for each τ, $\mathscr{S}(\tau) \cup \{\mathscr{I}^+ \cap J^-(\mathscr{Q}(\tau))\}$ is a Cauchy surface in $\bar{M}$ for $D^+(\mathscr{S})$.

If there is a future event horizon $\partial J^-(\mathscr{I}^+)$ in the region $D^+(\mathscr{S})$ of a future asymptotically predictable space, then it follows from Property 2 above that the surfaces $\mathscr{S}(\tau)$ will intersect the event horizon for sufficiently large τ, so $\mathscr{B}(\tau) \equiv \mathscr{S}(\tau) \setminus J^-(\mathscr{I}^+)$ will be non-empty. A *black hole* on the surface $\mathscr{S}(\tau)$ is a connected component of $\mathscr{B}(\tau)$.

[4]This condition, called the *null convergence condition*, is implied by the *weak energy condition* which says that the energy density measured by any observer is non-negative, that is, $T_{ab}W^aW^b \geq 0$, for any timelike vector $W^a \in T_p(M)$

3.2.3 The black string: a five-dimensional extension

Above, we gave a definition of the term "black hole" which applies to spacetimes that are weakly asymptotically simple and empty, however, this definition does not capture all spacetimes that we would like to regard as black holes. For example, consider the $(4+1)$-dimensional vacuum spacetime equipped with the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + d\xi^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.55}$$

This spacetime is referred to as a "black string;" it is simply the Schwarzschild geometry augmented by one more spacelike dimension. Clearly it is a black object of some kind, as the addition of one more spatial dimension does not substantively change the properties of the solution. Nevertheless, in computing the conformal rescaling of the metric, one expects $\Omega \sim 1/r + O(1/r^2)$ [43], which results in the induced metric on $\mathscr{I}$ becoming singular. One can see this easily by switching to Ω in favour of the r co-ordinate in the rescaled metric, where it becomes evident that the induced metric on the surface $\Omega = 0$ is such that the leading orders in r of the metric coefficients must at least match to avoid a singular induced metric on the boundary. This degeneracy means that we cannot attach a proper codimension 1 boundary to the metric by conformal completion, and so Hawking's definition of a black hole seems to fail in this case. Despite that this metric is not weakly asymptotically simple and empty, it is nevertheless well-enough behaved that one still has the notion of "infinitely far away," even if it doesn't seem possible to attach a good boundary to the spacetime. Both of the new solutions presented in this thesis also suffer from this effect.

3.2.4 Black hole thermodynamics

In 1971, Hawking [14] proved the "black hole area theorem," which stated that under the conditions of a strongly asymptotically predictable spacetime satisfying the weak energy condition, that the total area A of the event horizons in a spacetime never decreases; that is, $\delta A \geq 0$. It was noticed that this resembled the second law of thermodynamics; that for the total entropy S of a system, $\delta S \geq 0$. Furthermore, this correspondence could be extended, at least formally, to analogous statements for black holes for each of the four laws of thermodynamics. This correspondence is detailed in Table 3.2. The analogous quantities are entropy S with $A/8\pi\alpha$, temperature T with $\alpha\kappa$, and energy E with mass M. The quantity κ is the surface gravity of the black hole, and α is a proportionality constant. This correspondence was initially regarded as purely formal; in the classical theory, black holes are perfect absorbers and do not emit anything, so their "temperature" should be regarded as absolute zero. How, then, did κ correspond to a physical temperature? In

Law	Thermodynamics	Black Holes
Zeroth	T is constant throughout a body in thermal equilibrium	κ constant over the horizon of a stationary black hole
First	$dE = TdS +$ work terms	$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ$
Second	$\delta S \geq 0$ in any process	$\delta A \geq 0$ in any process
Third	Impossible to achieve $T = 0$ by any physical process	Impossible to achieve $\kappa = 0$ by any physical process

Table 3.2: Laws of Thermodynamics compared to Black Holes. Based on Table 12.1 of [43].

1975, Hawking [16] employed the theory of quantum fields on a curved spacetime to show that black holes *did* radiate with a perfect Maxwellian energy distribution of temperature $T_H = \hbar\kappa/2\pi k_B$.

From arguments about the thermal Green's functions due to Gibbons and Perry in [13], for an $(n+2)$-dimensional *static* black hole one can calculate the Hawking temperature T_H by studying the Euclideanised metric for the solution under consideration. We start with the Euclideanised $(t \to it)$ metric

$$ds^2 = V(r)dt^2 + dr^2/V(r) \oplus d\bar{s}_n^2, \tag{3.56}$$

and expand $V(r)$ about the horizon $r = r_0$, $V(r) \approx (r - r_0)V'(r_0)$. This gives us

$$ds^2 = (r - r_0)V'(r_0)dt^2 + \frac{dr^2}{(r - r_0)V'(r_0)} \oplus d\bar{s}_n^2. \tag{3.57}$$

Introduce a new co-ordinate $z = (r - r_0)$ to obtain

$$ds^2 = zV'(r_0)dt^2 + \frac{dz^2}{zV'(r_0)} \oplus d\bar{s}_n^2, \tag{3.58}$$

and make a further co-ordinate change given by

$$y = \int dy = \int \frac{dz}{\sqrt{zV'(r_0)}} = 2\sqrt{\frac{z}{V'(r_0)}} \tag{3.59}$$

so $z = \frac{V'(r_0)}{4}y^2$, and

$$ds^2 = \frac{V'(r_0)^2 y^2}{4}dt^2 + dy^2 \oplus d\bar{s}_n^2. \tag{3.60}$$

One identifies the (t, y) part of the above metric with the metric on the plane, $dr^2 + r^2 d\theta^2$. Thus, t must be periodically identified with a period of $4\pi/V'(r_0)$, and this period is found

to be proportional to the inverse of the Hawking temperature. Thus, $T_H = V'(r_0)/4\pi$, in appropriate units. Note that this calculation is valid only when $V'(r_0) \neq 0$, that is, in cases where there is not a double root of V at the horizon. We will be calculating the Hawking temperatures for each of the black holes under consideration in the next chapter. In the Schwarzschild case, $T_H = 1/8\pi M$; in conventional units, the temperature of a black hole of M solar masses is found to be $6 \times 10^{-8}/M$ Kelvin [43], which would be extremely small for black holes expected to be formed by stellar collapse. The energy that is radiated is obtained from the mass of the black hole; in theory, black holes could "evaporate" via this process. For astrophysically realistic values of M, however, the expected lifetime of a black hole radiating at this rate can be shown to be many times longer than the current age of the universe.

Chapter 4

Black hole solutions

4.1 Introduction and motivation

Prior to the early 1990s, it was widely believed that in four dimensions, black holes should have a horizon with spherical topology. Hawking [15] proved that this was the case when one assumes asymptotic flatness and the dominant energy condition.[1] In the early 1990s, it became clear that in fact there are black hole solutions in four-dimensional general relativity which do not have a spherical topology: extending the 2+1 dimensional black hole of Bañados, Teitelboim, and Zanelli (the so-called BTZ black hole) [4][3], Lemos [27] gave a solution for a cylindrical black hole in four dimensions. A further generalisation of the work of Bañados *et al.* by Åminneborg, Bengtsson, Holst, and Peldán [1] discovered four-dimensional solutions which had the usual causal structure for black holes, but had event horizons with the topology of a Riemann surface of arbitrary genus. There was further work in this area done by Mann [29], and Brill [7], which further generalised the possibilities in four dimensions. Finally, Hawking's black hole topology theorem was generalised to higher dimensions by Cai and Galloway [9].

More than being an interesting problem in its own right, studying black hole solutions in *five* dimensions has become physically interesting because they may carry implications in certain physical theories; for example in Kaluza-Klein theory, quantum gravity, string theory, and the recent work surrounding "braneworlds." In fact, the new solution with horizons modeled on solvegeometry presented in this thesis is used to create a new example of a braneworld in [8], and Shiromizu and Ida [38] have mentioned this solution in connection with AdS-CFT correspondence. One also hopes to gain insight into which features of four dimensional black hole solutions remain in higher dimensions. Thurston's geometrisation conjecture (see Section 2.6) states that any compact, orientable, three-manifold M can be

[1] This condition is that for w_a timelike, $T^{ab}w_a w_b \geq 0$ and $T^{ab}w_a$ is not spacelike.

cut by disjoint embedded two-spheres and tori into pieces which are modeled on one of the eight model geometries. With the goal of providing some other examples of five-dimensional black holes with new horizon topologies, it seems reasonable to ask whether one can find a static five-dimensional vacuum solution with a black hole whose event horizon is modeled on a given geometry. In particular, we will seek solutions obeying the following set of assumptions:

1. The spacetime is a static vacuum solution to Einstein's field equations, and
2. The time-symmetric hypersurfaces of the spacetime are foliated by homogeneous three-surfaces generated by isometries of the spacetime.

In this chapter, we will use the convention that Greek indices (α, β, γ, etc.) will take all values from 0 to 4, and Latin indices (i, j, k) are specific to the three-surfaces foliating the time-symmetric hypersurfaces, and take on the values from 2 to 4. For simplicity, we will call these three-surfaces "the orbits." We will also employ a primed Latin index (i', j', k') to indicate any spatial value; from 1 to 4.

Let us now formulate the problem to be solved. The notion of a "static" spacetime is defined in, for example, Chapter 6 of Wald [43]. A *stationary* spacetime admits a timelike Killing vector field ξ, whose orbits are timelike curves. These Killing vectors generate isometries which give the time translation symmetry of the spacetime. For a *static* spacetime, one has the additional requirement that there exist spacelike surfaces Σ which are orthogonal to ξ at every point. These Killing vectors can be used to give a time co-ordinate on the spacetime manifold: a point $(t, \mathbf{x})$, where $\mathbf{x} = x^{i'} \in \Sigma$, is simply the point $\mathbf{x}$ evolved a parameter length t along the orbit of the Killing vector field ξ. The static assumption yields that ξ is orthogonal to the spatial directions at any t. Altogether, this means that the metric we will be studying has the form

$$ds^2 = -U(\mathbf{x})dt^2 + h_{i'j'}(\mathbf{x})dx^{i'}dx^{j'}, \tag{4.1}$$

with $h_{i'j'}$ a symmetric positive-definite 4×4 matrix.

We have also required that the spacetime is foliated by closed three-surfaces. So if we fix a value of t, then any point on the spacelike surface Σ_t lies upon a unique leaf of the foliation. One can simply define a smooth vector field which is orthogonal to each of the tangent vectors to the (three-dimensional) leaf at that point, and use the flows along this vector field to give us a new co-ordinate R. With both the static assumption and the existence of this foliation, one has a metric of the form

$$ds^2 = -U(\mathbf{x})dt^2 + W(\mathbf{x})dR^2 + h_{ij}(\mathbf{x})dx^i dx^j. \tag{4.2}$$

We have required that the leaves of the foliation be homogeneous, and generated by isometries of the spacetime. This means that the induced metric on the $t = \text{const.}$, $r = \text{const.}$ surfaces is invariant under the action of some group of isometries of the spacetime acting transitively on the surfaces. Thus, we should write the induced metric in the form $d\bar{s}_3^2 = h_{ij}(R)\omega^i\omega^j$, where the ω^i are the family of invariant one-forms on these surfaces. Because this group of isometries are isometries of the entire spacetime, the functions U and W should only depend on R. Changing co-ordinates with

$$r = \int \sqrt{U(R)W(R)}dR,$$

and defining $V(r) = U(R(r))$, one obtains the final metric

$$ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + h_{ij}(r)\omega^i\omega^j. \tag{4.3}$$

Note that the isometries of the three-manifolds discussed in Chapter 2 can easily be extended to isometries of the entire spacetime. If g is an isometry that preserves the invariant one-forms on the horizon manifold, we can simply extend g to an isometry g' of the entire spacetime by requiring that g does not change the t and r co-ordinates. That is, if $g(x, y, z) = (x', y', z')$, then $g'(t, r, x, y, z) = (t, r, x', y', z')$ is an isometry of the spacetime manifold. We can therefore use the manifolds considered in Chapter 2 as our candidates for the orbits. Factoring the orbits by a discrete cocompact subgroup of isometries can be performed after these manifolds have been embedded in the spacetime.

Denote by X^i and ω^i the invariant vector fields and one-forms on a homogeneous three-manifold, respectively. Using the conventions found in Ryan and Shepley [35], $X_i \cdot X_j = h_{ij}(r)$, $[X_i, X_j] = -C^s_{ij}X_s$, and $d\omega^i = \frac{1}{2}C^i_{st}\omega^s \wedge \omega^t$, where C^i_{jk} are the structure constants of the isometry group of the orbits. The values of the structure constants and co-ordinate representations of the invariant one-forms are given in [35], and some are reproduced here in Table 2.1. The zero set of $V(r)$ will be the horizon for our black holes.

For a vacuum solution, we require that the stress-energy tensor vanish. So Einstein's equations become

$$\begin{aligned} G_{\alpha\beta} + \Lambda g_{\alpha\beta} &= 0 \\ R_{\alpha\beta} + \left(\Lambda - \frac{1}{2}R\right) g_{\alpha\beta} &= 0. \end{aligned}$$

Contracting with $g^{\alpha\beta}$, and simplifying, we have that (in five dimensions) $R = 10\Lambda/3$ for a vacuum solution. Einstein's equations become:

$$R_{\alpha\beta} - \frac{2}{3}\Lambda g_{\alpha\beta} = 0. \tag{4.4}$$

Any spacetime which is a vacuum solution is thus an Einstein manifold.

Several solutions of this type under consideration are already known; see for example the work of Birmingham [6]. Of the eight model geometries, solutions with a horizon modeled on nilgeometry, solvegeometry, and the geometry of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ have not yet appeared. We will review the known solutions, and present new solutions for each of nilgeometry and solvegeometry. The $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ case remains open; the system of equations remaining to be solved (with the further assumption that $h_{ij}(r)$ be diagonal) is displayed.

4.2 Calculation of the five-dimensional Ricci curvature

In this section, we ultimately wish to compute the Ricci curvature of this metric, in order to write the equations for a vacuum solution. This calculation parallels, but is not identical to, a four-dimensional non-static case considered by Ryan and Shepley in Chapter 9 of [35]. First we change to an orthonormal basis:

$$\begin{aligned}
\sigma^0 &= \sqrt{V}dt \\
\sigma^1 &= \frac{dr}{\sqrt{V}} \\
\sigma^i &= b_{is}(r)\omega^s,
\end{aligned}$$

where $b_{is}b_{sj} = g_{ij}$, and $b_{ij} = b_{ji}$. b_{ij} is the symmetric square root of the symmetric matrix h_{ij}. In this basis,

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu. \tag{4.5}$$

Now employ the Misner decomposition, and write $(\det B)^{1/3} = e^{-\Omega(r)}$, for a function Ω. Then $e^{\Omega}b_{ij} = e^{\beta(r)}_{ij}$, where β_{ij} is a 3×3 traceless matrix, and because we are using matrix exponentiation it follows that $e^{\beta(r)}_{ij}$ has unit determinant. To compute the connection one-forms $\sigma_{\mu\nu}$, we first need to compute the exterior derivatives of the basis one-forms. They are:

$$\begin{aligned}
d\sigma^0 &= \frac{V'}{2\sqrt{V}}dr\wedge dt \\
&= \frac{V'}{2\sqrt{V}}\sigma^1\wedge\sigma^0 \\
d\sigma^1 &= 0 \\
d\sigma^i &= d(e^{-\Omega}e^{\beta}_{is}\omega^s) \\
&= (-\Omega' e^{-\Omega}e^{\beta}_{is} + e^{-\Omega}(e^{\beta}_{is})')dr\wedge\omega^s + e^{-\Omega}e^{\beta}_{is}d\omega^s \\
&= (-\Omega'\delta_{iu} + (e^{\beta}_{it})'e^{-\beta}_{tu})e^{-\Omega}e^{\beta}_{us}dr\wedge\omega^s + \frac{1}{2}e^{-\Omega}e^{\beta}_{is}C^s_{tu}\omega^t\wedge\omega^s \\
&= \sqrt{V}(-\Omega'\delta_{iu} + (e^{\beta}_{it})'e^{-\beta}_{tu})\sigma^1\wedge\sigma^u + \frac{1}{2}e^{\Omega}e^{\beta}_{is}C^s_{tu}e^{-\beta}_{tj}e^{-\beta}_{uk}\sigma^j\wedge\sigma^k.
\end{aligned}$$

Define $k_{ij} = \sqrt{V}(-\Omega'\delta_{iu} + (e^{\beta}_{it})' e^{-\beta}_{tu})$, and $d^i_{jk} = e^{\Omega} e^{\beta}_{is} C^s_{tu} e^{-\beta}_{tj} e^{-\beta}_{uk}$. Notice that the d^i_{jk} inherit the anti-symmetry of the C^i_{jk} on interchange of the lower two indices. Then we have

$$d\sigma^i = k_{ij}\sigma^1 \wedge \sigma^j + \frac{1}{2} d^i_{jk}\sigma^j \wedge \sigma^k.$$

Using the compatibility of the metric with the covariant derivative we have $d\eta_{\mu\nu} = 0 = \sigma_{\mu\nu} + \sigma_{\nu\mu}$, so there are only 10 independent connection one-forms to be found. This relationship gives the following symmetries on the connection one-forms:

$$\begin{aligned} \sigma_{\mu\mu} &= 0 \\ \sigma^{i'}{}_0 &= \sigma^0{}_{i'} \\ \sigma^{i'}{}_{j'} &= -\sigma^{j'}{}_{i'}. \end{aligned}$$

These symmetries give rise to symmetries on the connection coefficients $\Gamma^{\mu}_{\nu\gamma}$, defined by $\sigma^{\mu}{}_{\nu} = \Gamma^{\mu}_{\nu\gamma}\sigma^{\gamma}$:

$$\begin{aligned} \Gamma^{i'}_{0\gamma} &= \Gamma^0_{i'\gamma} \\ \Gamma^{i'}_{j\gamma} &= -\Gamma^j_{i'\gamma}. \end{aligned}$$

We proceed by solving Cartan's first equation

$$0 = d\sigma^{\mu} + \sigma^{\mu}{}_{\nu} \wedge \sigma^{\nu}$$

for the connection one-forms. For the $\mu = 0$ equation, this immediately leads to

$$\sigma^0{}_1 = \sigma^1{}_0 = \frac{V'}{2\sqrt{V}}\sigma^0 \tag{4.6}$$

$$\sigma^0{}_i = \sigma^i{}_0 = 0. \tag{4.7}$$

Continuing to the $\mu = 1$ equation, we have

$$0 = \sigma^1{}_i \wedge \sigma^i, \tag{4.8}$$

which will shall require later. Using Equation 4.7 and expanding the sum, the remaining $\mu = i$ equations are:

$$0 = k_{ij}\sigma^1 \wedge \sigma^j + \frac{1}{2} d^i_{jk}\sigma^j \wedge \sigma^k + \sigma^i{}_1 \wedge \sigma^1 + \sigma^i{}_j \wedge \sigma^j. \tag{4.9}$$

Introducing the connection coefficients, Equations 4.8 and 4.9 become

$$0 = \Gamma^1_{i\gamma}\sigma^{\gamma} \wedge \sigma^i \tag{4.10}$$

$$0 = k_{ij}\sigma^1 \wedge \sigma^j + \frac{1}{2} d^i_{jk}\sigma^j \wedge \sigma^k + \Gamma^i_{1\gamma}\sigma^{\gamma} \wedge \sigma^1 + \Gamma^i_{j\gamma}\sigma^{\gamma} \wedge \sigma^j. \tag{4.11}$$

From Equation 4.11, we have that $\Gamma^i_{10} = \Gamma^1_{i0} = 0$ and $\Gamma^i_{j0} = 0$ in order that all terms in the expression will cancel. Similarly, from Equation 4.10 we have that $\Gamma^1_{i1} = \Gamma^i_{11} = 0$. We can then rewrite Equations 4.10 and 4.11 as:

$$\begin{aligned} 0 &= \Gamma^1_{ij}\sigma^j \wedge \sigma^i & (4.12)\\ \text{and}\quad 0 &= k_{ij}\sigma^1 \wedge \sigma^j + \frac{1}{2}d^i_{jk}\sigma^j \wedge \sigma^k + \Gamma^i_{1j}\sigma^j \wedge \sigma^1 + \Gamma^i_{j1}\sigma^1 \wedge \sigma^j + \Gamma^i_{jk}\sigma^k \wedge \sigma^j \\ &= k_{ij}\sigma^1 \wedge \sigma^j + \frac{1}{2}d^i_{jk}\sigma^j \wedge \sigma^k + (\Gamma^i_{1j} - \Gamma^i_{j1})\sigma^j \wedge \sigma^1 + \Gamma^i_{jk}\sigma^k \wedge \sigma^j. & (4.13)\end{aligned}$$

Equation 4.12 gives us that Γ^1_{ij} is symmetric on the lower two indices, and Equation 4.13 says that $k_{ij} = \Gamma^i_{1j} - \Gamma^i_{j1}$ and $d^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$. Keeping the required symmetries in mind, we can solve for all of the connection coefficients. Tables 4.1 and 4.2 show the resulting non-zero connection coefficients and connection one-forms, respectively. Note that in Table 4.1, that a symmetrisation $l_{ij} = -(k_{ij} + k_{ji})/2$ and anti-symmetrisation $m_{ij} = (k_{ij} - k_{ji})/2$ of k_{ij} is defined. The quantity l_{ij} can be interpreted as the second fundamental form (or equivalently the shape tensor) of the orbits as submanifolds of the spacetime. The *second fundamental form* is the tensor N defined by $N(X,Y) = (\nabla_X Y)^\perp$, where X, Y are vectors tangent to the submanifold, the covariant derivative ∇ is the one on the entire spacetime, and the $\perp$ symbol indicates that we take the component of the result that is normal to the submanifold. If we take the basis $\{Y_\mu\}$ dual to the basis of one-forms $\{\sigma^\mu\}$, we can compute the components of this tensor as follows:

$$\begin{aligned} N_{ij} = N(Y_i, Y_j) &= (\nabla_{Y_i} Y_j)^\perp \\ &= (\Gamma^\sigma_{ij} Y_\sigma)^\perp \\ &= (\Gamma^0_{ij} Y_0 + \Gamma^1_{ij} Y_1 + \Gamma^k_{ij} Y_k)^\perp \\ &= \Gamma^1_{ij} = l_{ij}, \end{aligned}$$

which gives the result as claimed.

We can now calculate the curvature two-forms $\mathscr{R}^\mu{}_\nu = d\sigma^\mu{}_\nu + \sigma^\mu{}_\alpha \wedge \sigma^\alpha{}_\nu$. From the curvature two-forms, the components of the Riemann tensor can be obtained via Cartan's second equation, $\mathscr{R}^\mu{}_\nu = R^\mu{}_{\nu|\alpha\beta|}\sigma^\alpha \wedge \sigma^\beta$, where we restrict the sum over α and β to $\alpha < \beta$, so that each index pair occurs only once. Using the components of the Riemann tensor, we can obtain the components of the Ricci tensor via the contraction $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$. Tables 4.3 and 4.4 give the resulting non-zero curvature two-forms and components of the Ricci tensor. The components of the Ricci tensor will be especially useful in Sections 4.3.1, 4.3.3, 4.3.4, and 4.3.5. Note that these are the components of the Ricci curvature in the orthonormal frame that we entered at the outset of this calculation; in order to obtain the components of the Ricci curvature in a co-ordinate frame, one must select a set of invariant one-forms

$$
\begin{aligned}
\Gamma^0_{10} = \Gamma^1_{00} &= \frac{V'}{2\sqrt{V}} \\
\Gamma^1_{ij} = -\Gamma^i_{1j} &= -\frac{1}{2}(k_{ij} + k_{ji}) := l_{ij} \\
\Gamma^i_{j1} &= \frac{1}{2}(k_{ji} - k_{ij}) := m_{ji} \\
\Gamma^i_{jk} &= \frac{1}{2}(d^i_{jk} - d^k_{ij} - d^j_{ik})
\end{aligned}
$$

Table 4.1: Non-zero connection coefficients for the five-dimensional metric

$$
\begin{aligned}
\sigma^0{}_1 = \sigma^1{}_0 &= \frac{V'}{2\sqrt{V}}\sigma^0 \\
\sigma^1{}_i = -\sigma^i{}_1 &= l_{ij}\sigma^j \\
\sigma^i{}_j &= \frac{1}{2}(d^i_{jk} - d^k_{ij} - d^j_{ik})\sigma^k - m_{ij}\sigma^1
\end{aligned}
$$

Table 4.2: Non-zero connection one-forms for the five-dimensional metric

ω^i in terms of the co-ordinate frame, and take the appropriate combination of curvature components.

4.3 Solutions

4.3.1 Warped products I: S^3, H^3, and E^3 solutions

Initially, it is reasonable to ask under what conditions a warped product of the form

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + f(r)^2\delta_{ij}\omega^i\omega^j, \tag{4.14}$$

will yield a vacuum solution. We can write this as the so-called "warped product" $ds^2 = -V(r)dt^2 + dr^2/V(r) \oplus f(r)^2 d\bar{s}^2$, where $d\bar{s}^2 = \delta_{ij}\omega^i\omega^j$. The Ricci curvature of the three-dimensional metric $d\bar{s}^2$ is found to be

$$\bar{R}_{ij} = \frac{1}{2}C^k_{kl}(C^i_{lj} + C^j_{li}) - \frac{1}{2}C^l_{ik}(C^k_{jl} + C^l_{jk}) + \frac{1}{4}C^i_{lk}C^j_{lk}. \tag{4.15}$$

Using the expressions found in Table 4.4, one finds that the curvature of the five-dimensional metric 4.14 can be written as

$$R_{00} = \frac{1}{2}\left(V'' + 3V'\frac{f'}{f}\right)$$

$$
\begin{aligned}
\mathscr{R}^0{}_1 = \mathscr{R}^1{}_0 &= \frac{V''}{2}\sigma^1 \wedge \sigma^0 \\
\mathscr{R}^0{}_i = \mathscr{R}^i{}_0 &= \frac{V'}{2\sqrt{V}} l_{ij}\sigma^0 \wedge \sigma^j \\
\mathscr{R}^1{}_i = -\mathscr{R}^i{}_1 &= (\sqrt{V} l'_{ik} + l_{ij}k_{kj} + l_{jk}m_{ji})\sigma^1 \wedge \sigma^k \\
&+ \frac{1}{2}(l_{ij}d^j_{kl} + l_{jk}(d^j_{il} - d^l_{ji} - d^i_{jl}))\sigma^k \wedge \sigma^l \\
\mathscr{R}^i{}_j &= \frac{1}{2}\Big[\sqrt{V}(d'^i_{jm} - d'^m_{ij} - d'^j_{im}) + k_{lm}(d^i_{jl} - d^l_{ij} - d^j_{il}) + m_{lj}(d^i_{lm} - d^m_{il} - d^l_{im}) \\
&- m_{il}(d^l_{jm} - d^m_{lj} - d^j_{lm})\Big]\sigma^1 \wedge \sigma^m + \Big[\frac{1}{4}d^l_{mn}(d^i_{jl} - d^l_{ij} - d^j_{il}) \\
&+ \frac{1}{4}(d^i_{lm} - d^m_{il} - d^l_{im})(d^l_{jn} - d^n_{lj} - d^j_{ln}) - l_{im}l_{jn}\Big]\sigma^m \wedge \sigma^n
\end{aligned}
$$

Table 4.3: Non-zero curvature two-forms for the five-dimensional metric

$$
\begin{aligned}
R_{00} &= \frac{1}{2}\left(V'' - \frac{V'}{\sqrt{V}} l_{ii}\right) \\
R_{11} &= -\frac{V''}{2} + \sqrt{V} l'_{ii} + l_{ij}k_{ji} + l_{ij}m_{ji} \\
R_{1i} &= -l_{jk}d^k_{ji} - l_{ki}d^j_{kj} \\
R_{ij} &= \frac{V'}{2\sqrt{V}} l_{ij} + \sqrt{V} l'_{ij} + l_{ik}k_{kj} - l_{kj}k_{ik} - l_{kk}l_{ij} + \frac{1}{2}d^k_{kl}(d^i_{lj} + d^j_{li}) \\
&- \frac{1}{2}d^l_{ik}(d^k_{jl} + d^l_{jk}) + \frac{1}{4}d^i_{lk}d^j_{lk}
\end{aligned}
$$

Table 4.4: Non-zero components of the Ricci tensor for the five-dimensional metric

$$
\begin{aligned}
R_{11} &= -\frac{1}{2}\left(V'' + 3V'\frac{f'}{f}\right) - 3V\frac{f''}{f} \\
R_{ij} &= -\left(V'\frac{f'}{f} + V\frac{f''}{f} - 2V\left(\frac{f'}{f}\right)^2\right)\delta_{ij} + \frac{1}{f^2}\bar{R}_{ij},
\end{aligned}
$$

with the other components vanishing. Using the field equations 4.4, we obtain the following for a vacuum solution:

$$
\begin{aligned}
\frac{1}{2}\left(V'' + 3V'\frac{f'}{f}\right) &= -\frac{2\Lambda}{3} && (4.16)\\
f'' &= 0 && (4.17)\\
-\left(V'\frac{f'}{f} - 2V\left(\frac{f'}{f}\right)^2\right)\delta_{ij} + \frac{1}{f^2}\bar{R}_{ij} &= \frac{2\Lambda}{3}\delta_{ij}. && (4.18)
\end{aligned}
$$

Equation 4.17 implies that $f(r) = ar + b$, with a and b constants. It suffices to consider the cases $f(r) = 1$ and $f(r) = r$, as the general case can be reduced to one of these via co-ordinate transformations.

Suppose $f(r) = r$. Equation 4.16 gives that $V(r) = -\Lambda r^2/6 + k - 2M/r^2$, with k and M constants. With this $V(r)$, Equation 4.18 becomes $\bar{R}_{ij} = 2k\delta_{ij}$. This says that the three-dimensional manifold must be Einstein. We can thus obtain solutions with horizon topology H^3 ($k = -1$), E^3 ($k = 0$), and S^3 ($k = 1$). These solutions appear in Birmingham [6], and are five-dimensional versions of the Schwarzschild solution ($\Lambda = 0, k = 1$), Kottler solution ($\Lambda < 0, k = 1$), "hyperbolic Kottler" solutions ($\Lambda < 0, k = -1$)[1][29], and Lemos solutions ($\Lambda < 0, k = 0$) [27].

Suppose $f(r) = 1$. Then from Equation 4.16, we have that $V(r) = -2\Lambda r^2/3 + Ar + C$, with A and C constants. Here we can put $A = 0$ without loss of generality, by employing the co-ordinate transformation $R = r - 3A/4\Lambda$. Equation 4.18 becomes $\bar{R}_{ij} = \frac{2\Lambda}{3}\delta_{ij}$, which says that for a vacuum solution the three-dimensional manifold must again be Einstein. These are five-dimensional analogues to the four-dimensional Nariai solution, and have they have the same topologies as Birmingham's solutions, above.

To be explicit, the line elements for these solutions are: For H^3,

$$
\begin{aligned}
ds^2 &= -\left(-\frac{\Lambda}{6}r^2 - 1 - \frac{2M}{r^2}\right)dt^2 + \frac{dr^2}{-\frac{\Lambda}{6}r^2 - 1 - \frac{2M}{r^2}} \\
&\quad + r^2(dx^2 + \sinh^2(x)dy^2 + \sinh^2(x)\sin^2(y)dz^2), \text{ and} && (4.19)\\
ds^2 &= -\left(-\frac{2\Lambda}{3}r^2 + C\right)dt^2 + \frac{dr^2}{-\frac{2\Lambda}{3}r^2 + C} \\
&\quad - \frac{3}{\Lambda}(dx^2 + \sinh^2(x)dy^2 + \sinh^2(x)\sin^2(y)dz^2). && (4.20)
\end{aligned}
$$

For E^3, the $f(r) = 1$ case becomes the five-dimensional Minkowski metric. For $f(r) = r$,

we have the solution

$$ds^2 = -\left(-\frac{\Lambda}{6}r^2 - \frac{2M}{r^2}\right) + \frac{dr^2}{-\frac{\Lambda}{6}r^2 - \frac{2M}{r^2}} + r^2(dx^2 + dy^2 + dz^2). \tag{4.21}$$

Finally, for S^3 we have the solutions

$$\begin{aligned} ds^2 &= -\left(-\frac{\Lambda}{6}r^2 + 1 - \frac{2M}{r^2}\right) dt^2 + \frac{dr^2}{-\frac{\Lambda}{6}r^2 + 1 - \frac{2M}{r^2}} \\ &\quad + r^2(dx^2 + \sin^2(x)dy^2 + \sin^2(x)\sin^2(y)dz^2), \text{ and} \qquad (4.22) \\ ds^2 &= -\left(-\frac{2\Lambda}{3}r^2 + C\right) dt^2 + \frac{dr^2}{-\frac{2\Lambda}{3}r^2 + C} \\ &\quad + \frac{3}{\Lambda}(dx^2 + \sin^2(x)dy^2 + \sin^2(x)\sin^2(y)dz^2). \qquad (4.23) \end{aligned}$$

4.3.2 Warped products II: $S^1 \times H^2$ and $S^1 \times S^2$ solutions

In this section, we are seeking solutions whose horizons are products of lower dimensional manifolds. One way of obtaining examples of solutions with these horizon topologies is to consider warped products of black hole solutions in lower dimensions with horizons that are a factor of the desired horizon manifold in five dimensions. Two such solutions are the aforementioned BTZ black hole [4][3] which has a horizon with S^1 topology, and is given by the metric

$$ds^2 = -(-\Lambda r^2 - M)dt^2 + \frac{dr^2}{-\Lambda r^2 - M} + r^2 d\xi^2, \tag{4.24}$$

in the zero angular momentum case. Here, $\Lambda < 0$. Another such solution is the Schwarzschild solution, discussed in Section 3.2.1.

Let us begin by first considering the BTZ solution, and the ansatz $ds^2 = d\bar{s}_3^2 \oplus f(r)d\tilde{s}_2^2$, where $d\bar{s}_3^2$ is the metric on the three-dimensional BTZ black hole, and $d\tilde{s}_2^2$ is the two-dimensional metric on either H^2 or S^2. One finds that the following solutions can be obtained: If we put $d\tilde{s}_2^2$ to be the metric on H^2, we obtain that $f(r) = 1$ and

$$\begin{aligned} ds^2 &= -\left(-\frac{\Lambda}{3}r^2 - M\right) dt^2 + \frac{dr^2}{-\frac{\Lambda}{3}r^2 - M} \\ &\quad + r^2 d\xi^2 - \frac{3}{2\Lambda}(d\theta^2 + \sinh^2(\theta)d\phi^2) \qquad (4.25) \end{aligned}$$

is a solution for $\Lambda < 0$ with a horizon located at $r = \sqrt{3M/\Lambda}$ when $M > 0$.

Attempting this when $d\tilde{s}_2^2$ is the metric on S^2, we obtain that

$$\begin{aligned} ds^2 &= -\left(-\frac{\Lambda}{3}r^2 + C\right) dt^2 + \frac{dr^2}{-\frac{\Lambda}{3}r^2 + C} \\ &\quad + r^2 d\xi^2 + \frac{3}{2\Lambda}(d\theta^2 + \sin^2(\theta)d\phi^2) \qquad (4.26) \end{aligned}$$

solves the field equations when $\Lambda > 0$, however it is not a black hole exterior solution. For $C > 0$, $V(r) = -\frac{\Lambda}{3}r^2 + C$ is positive between $-\sqrt{3C/\Lambda} < r < \sqrt{3C/\Lambda}$. For $C \leq 0$, the co-ordinate r is interpreted as the time co-ordinate, and the solution is not static. In fact, this case *must* fail, as the analogue of 4.18 for this case implies that $\Lambda > 0$ for the S^2 factor. Ida [20] proved that a three-dimensional spacetime is a black hole only if $\Lambda < 0$, which is a contradiction, so this ansatz fails.

Next we consider the Schwarzschild solution for a $S^2 \times S^1$ solution. By simply appending a $d\xi^2$ term to the metric, one finds that a five-dimensional solution with $S^2 \times S^1$ horizons is obtained. This solution is sometimes called a "black string" solution, and the metric is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + d\xi^2 \\ &\quad + r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \end{aligned} \tag{4.27}$$

This yields a solution when $\Lambda = 0$, with the horizon located at $r = 2M$.

4.3.3 Nilgeometry

If we take the ω^i to be the Bianchi II invariant one-forms, a solution will have its horizon and exterior region modeled on nilgeometry. Referring to Table 2.1, we have the following structure constants and invariant one-forms:

$$\begin{aligned} C^2_{34} = -C^2_{43} &= 1 \\ \omega^2 &= dy - xdz \\ \omega^3 &= dz \\ \omega^4 &= dx. \end{aligned}$$

Taking h_{ij} to be diagonal, with $h_{22} = f(r)$, $h_{33} = g(r)$ and $h_{44} = h(r)$, we obtain the following system from Equation 4.4 and Table 4.4:

$$2V'' + V'\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right) = -\frac{8\Lambda}{3} \tag{4.28}$$

$$2\left(\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h}\right) - \left[\left(\frac{f'}{f}\right)^2 + \left(\frac{g'}{g}\right)^2 + \left(\frac{h'}{h}\right)^2\right] = 0 \tag{4.29}$$

$$-2V'\frac{f'}{f} + 2V\left[\left(\frac{f'}{f}\right)^2 - \frac{f''}{f} - \frac{f'}{2f}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] + \frac{2f}{gh} = \frac{8\Lambda}{3} \tag{4.30}$$

$$-2V'\frac{g'}{g} + 2V\left[\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{g'}{2g}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] - \frac{2f}{gh} = \frac{8\Lambda}{3} \tag{4.31}$$

$$-2V'\frac{h'}{h} + 2V\left[\left(\frac{h'}{h}\right)^2 - \frac{h''}{h} - \frac{h'}{2h}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] - \frac{2f}{gh} = \frac{8\Lambda}{3}. \tag{4.32}$$

These equations are solved by taking $V(r) = -2\Lambda r^2/11 - 2M/r^{5/3}$, $f(r) = r^{8/3}$, $g(r) = -9r^{4/3}/4\Lambda$, and $h(r) = r^{4/3}$. After re-scaling and re-labelling the co-ordinates, the metric

$$\begin{aligned} ds^2 &= -\left(-\frac{2\Lambda}{11}r^2 - \frac{2M}{r^{5/3}}\right)dt^2 + \frac{dr^2}{-\frac{2\Lambda}{11}r^2 - \frac{2M}{r^{5/3}}} \\ &\quad + r^{4/3}(dx^2 + dy^2) + r^{8/3}\left(dz - \sqrt{\frac{-4\Lambda}{9}}x dy\right)^2 \end{aligned} \tag{4.33}$$

is a solution with $\Lambda < 0$, and horizon located at $r = (-11M/\Lambda)^{3/11}$. This solution has Kretschmann scalar

$$I = \frac{4}{3267}\frac{1709\Lambda^2 r^{22/3} + 133584M^2 + 2992\Lambda M r^{11/3}}{r^{22/3}},$$

and Hawking temperature $T_H = (11\Lambda^{8/3}M)^{3/11}/6\pi$.

The covering space for this solution has another family of isometries: mapping $(t,r,x,y,z) \mapsto (t/a, ar, x/a^{2/3}, y/a^{2/3}, z/a^{4/3})$ is an isometry between solutions with parameter value M and $M/a^{11/3}$. This kind of rescaling isometry is not present in the Schwarzschild solution, but is present in the negative mass solutions considered by Horowitz and Myers [18], and is an indication that the parameter M is not trivially related to the mass of the black hole. It should also be noted that this isometry of the covering spaces does not preserve the identifications made to compactify the three-manifolds foliating the event horizon.

4.3.4 Solvegeometry

The solvegeometry case corresponds to Bianchi VI_{-1}. Referring to Table 2.1, we have the following structure constants and invariant one-forms:

$$\begin{aligned} C^2_{24} = -C^2_{42} &= 1 \\ C^3_{34} = -C^3_{43} &= -1 \\ \omega^2 &= e^{-x}dy \\ \omega^3 &= e^{x}dz \\ \omega^4 &= dx. \end{aligned}$$

If we take h_{ij} to be diagonal, with $h_{22} = f(r)$, $h_{33} = g(r)$, and $h_{44} = h(r)$, we obtain the following system from Equation 4.4 and Table 4.4:

$$2V'' + V'\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right) = -\frac{8\Lambda}{3} \tag{4.34}$$

$$2\left(\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h}\right) - \left[\left(\frac{f'}{f}\right)^2 + \left(\frac{g'}{g}\right)^2 + \left(\frac{h'}{h}\right)^2\right] = 0 \tag{4.35}$$

$$-2V'\frac{f'}{f} + 2V\left[\left(\frac{f'}{f}\right)^2 - \frac{f''}{f} - \frac{f'}{2f}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] = \frac{8\Lambda}{3} \tag{4.36}$$

$$-2V'\frac{g'}{g} + 2V\left[\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{g'}{2g}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] = \frac{8\Lambda}{3} \tag{4.37}$$

$$-2V'\frac{h'}{h} + 2V\left[\left(\frac{h'}{h}\right)^2 - \frac{h''}{h} - \frac{h'}{2h}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] - \frac{8}{h} = \frac{8\Lambda}{3} \tag{4.38}$$

One finds that these equations are satisfied by $V(r) = -2\Lambda r^2/9 - 2M/r$, $f(r) = g(r) = r^2$, and $h(r) = -3/\Lambda$. After a re-scaling and re-labelling of the co-ordinates, the solution can be written as:

$$\begin{aligned} ds^2 &= -\left(-\frac{2\Lambda}{9}r^2 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{-\frac{2\Lambda}{9}r^2 - \frac{2M}{r}} \\ &\quad + \left(\frac{3}{-\Lambda}\right)\left(r^2\left(e^{2z}dx^2 + e^{-2z}dy^2\right) + dz^2\right), \end{aligned} \tag{4.39}$$

with $\Lambda < 0$ and horizon located at $r = (-9M/\Lambda)^{1/3}$. This solution has Kretschmann scalar

$$I = \frac{4}{9}\frac{7\Lambda^2 r^6 + 12\Lambda M r^3 + 108M^2}{r^6},$$

and Hawking temperature $T_H = (\Lambda^2 M/24\pi^3)^{1/3}$.

One also has isometries of covering spaces with similar scaling behaviour as the nilgeometry case: the map $(t, r, x, y, z) \mapsto (t/a, ar, x/a, y/a, z)$ is an isometry between the covering spaces of solutions with parameter M and M/a^3.

4.3.5 $\widetilde{\mathrm{SL}}(2, \mathbb{R})$

$\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is the universal cover of the group of 2×2 matrices with determinant 1, and this model geometry corresponds to Bianchi type VIII. $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ has the structure of a line bundle over the hyperbolic plane H^2, but is not isometric to the $H^2 \times \mathbb{R}$ case. A solution for the $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ case is not presented here, but the resulting field equations will be. Referring to Table 2.1, we have the following structure constants and invariant one-forms:

$$\begin{aligned} C^2_{34} = -C^2_{43} &= -1 \\ C^3_{42} = -C^3_{24} &= 1 \\ C^4_{23} = -C^4_{32} &= 1 \\ \omega^2 &= dx + (1 + x^2)dy + (x - y - x^2 y)dz \\ \omega^3 &= 2xdy + (1 - 2xy)dz \\ \omega^4 &= dx + (-1 + x^2)dy + (x + y - x^2 y)dz \end{aligned}$$

If we take h_{ij} to be diagonal, with $h_{22} = f(r)$, $h_{33} = g(r)$, and $h_{44} = h(r)$, we obtain the following system from Equation 4.4 and Table 4.4:

$$2V'' + V'\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right) = -\frac{8\Lambda}{3} \tag{4.40}$$

$$2\left(\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h}\right) - \left[\left(\frac{f'}{f}\right)^2 + \left(\frac{g'}{g}\right)^2 + \left(\frac{h'}{h}\right)^2\right] = 0 \tag{4.41}$$

$$\begin{aligned} -2V'\frac{f'}{f} + 2V\left[\left(\frac{f'}{f}\right)^2 - \frac{f''}{f} - \frac{f'}{2f}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] & \\ + \frac{2f}{gh} - \frac{2}{f}\left(\frac{g}{h} + \frac{h}{g} - 2\right) &= \frac{8\Lambda}{3} \end{aligned} \tag{4.42}$$

$$\begin{aligned} -2V'\frac{g'}{g} + 2V\left[\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{g'}{2g}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] & \\ + \frac{2g}{fh} - \frac{2}{g}\left(\frac{f}{h} + \frac{h}{f} + 2\right) &= \frac{8\Lambda}{3} \end{aligned} \tag{4.43}$$

$$\begin{aligned} -2V'\frac{h'}{h} + 2V\left[\left(\frac{h'}{h}\right)^2 - \frac{h''}{h} - \frac{h'}{2h}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)\right] & \\ + \frac{2h}{fg} - \frac{2}{h}\left(\frac{f}{g} + \frac{g}{f} + 2\right) &= \frac{8\Lambda}{3}. \end{aligned} \tag{4.44}$$

Note that this system is symmetric in $g(r)$ and $h(r)$, so if a solution is found with $g(r) \neq h(r)$, a second solution is automatically obtained by interchanging $g(r)$ with $h(r)$. A solution for these equations of correct signature appears to be difficult to find: the only solutions that were located had incorrect signature, with $-f(r) = g(r) = h(r) = Cr^2$ or $-f(r) = g(r) = h(r) = C$, with C a constant. For example, one can attempt to find a solution by employing the ansatz $f(r) = Ar^s$, $g(r) = Br^q$, and $h(r) = Cr^s$. This leads to the following two "solutions" of the field equations that have incorrect signature:

$$V(r) = -\frac{\Lambda}{6}r^2 - \frac{1}{4} - \frac{2M}{r^2} \tag{4.45}$$

$$-f(r) = g(r) = h(r) = r^2, \tag{4.46}$$

and

$$V(r) = -\frac{2\Lambda}{3}r^2 + C, \tag{4.47}$$

$$-f(r) = g(r) = h(r) = -\frac{3}{4\Lambda}. \tag{4.48}$$

Chapter 5

Conclusion

In this thesis, we have successfully presented examples of new five-dimensional black hole solutions which have their event horizons modeled on seven of the eight three-dimensional geometries. The solutions for nilgeometry, with the metric

$$\begin{aligned} ds^2 &= -\left(-\frac{2\Lambda}{11}r^2 - \frac{2M}{r^{5/3}}\right)dt^2 + \frac{dr^2}{-\frac{2\Lambda}{11}r^2 - \frac{2M}{r^{5/3}}} \\ &\quad + r^{4/3}(dx^2 + dy^2) + r^{8/3}\left(dz - \sqrt{\frac{-4\Lambda}{9}}x dy\right)^2, \end{aligned}$$

and for solvegeometry, with the metric

$$\begin{aligned} ds^2 &= -\left(-\frac{2\Lambda}{9}r^2 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{-\frac{2\Lambda}{9}r^2 - \frac{2M}{r}} \\ &\quad + \left(\frac{3}{-\Lambda}\right)\left(r^2\left(e^{2z}dx^2 + e^{-2z}dy^2\right) + dz^2\right), \end{aligned}$$

were previously unknown. Because the horizons of these solutions are modeled on a given three-dimensional geometry, the horizons can be compactified by factoring by a cocompact subgroup of their isometry group. Like some of the previous solutions in five dimensions, these new solutions do not have a $\mathscr{I}$, although the event horizon still represents the boundary of the region from which timelike or null geodesics can escape to infinity. In theory, one could even pass to a Kruskal-like co-ordinate system to eliminate the co-ordinate pathology at the horizon, although the co-ordinate transformations are in general only known up to quadrature.

Yet to be found is a black hole with horizons modeled on the geometry of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, and the system of equations to be solved when one takes the metric to be diagonal is presented in Equations 4.40 through 4.44. It is not known whether a solution for this case exists, or

whether there is a fundamental obstruction to solutions having a horizon with this topology. Other new solutions might be sought by relaxing the condition that the event horizons are covered by a single three-dimensional geometry.

Future research could include the computation of mass for the new solutions as these solutions share a rescaling isometry of their covering spaces similar to the negative mass solutions of Horowitz and Myers [18], and this might provide insight into positive mass theorems. Other work might centre on using these five-dimensional solutions to study "braneworld" cosmologies (discussed in Randall and Sundrum [34], Krauss [25], and Ida [19]). The solve-geometry solution has lead to an example of a Bianchi VI braneworld [8][38].

Bibliography

[1] S. Åminneborg, I. Bengtsson, S. Holst, and P. Peldán. Making anti-de Sitter black holes. *Class. Quant. Grav.*, 13:2707–2714, 1996.

[2] A. Ashtekar and R. Hansen. Unified treatment of null and spatial infinity in general relativity. 1. Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity. *J. Math. Phys.*, 19(7):1542–1566, 1978.

[3] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli. Geometry of the 2+1 black hole. *Phys. Rev. D*, 48(4):1506–1525, 1993.

[4] M. Bañados, C. Teitelboim, and J. Zanelli. Black hole in three-dimensional spacetime. *Phys. Rev. Lett.*, 69(13):1849–1851, 1992.

[5] L. Bianchi. Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti. *Mem. Soc. It. Della. Sc. (Dei. XL)*, 11(3):267, 1897.

[6] D. Birmingham. Topological black holes in anti-de Sitter space. *Class. Quant. Grav.*, 16:1197–1205, 1999.

[7] D. Brill. Geometry of black holes and multi-black holes in 2+1 dimensions. *Helv. Phys. Acta*, 69:249–252, 1996.

[8] C. Cadeau and E. Woolgar. New five-dimensional black holes classified by horizon geometry, and a Bianchi VI braneworld. *Class. Quant. Grav.*, 18:527–542, 2001.

[9] M. Cai and G. Galloway. On the topology and area of higher dimensional black holes. `arXiv.org` e-Print hep-th/0102149, 2001. To appear in *Class. Quant. Grav.*

[10] S. Carroll. Lecture notes on general relativity. `arXiv.org` e-Print gr-qc/9712019, 1997.

[11] L. Conlon. *Differentiable manifolds: a first course.* Birkhäuser, 1993.

[12] H. Fagundes. Relativistic cosmologies with closed, locally homogeneous spatial sections. *Phys. Rev. Lett.*, 54(11):1200–1202, 1985.

[13] G. Gibbons and M. Perry. Black holes and thermal Green's functions. *Proc. Roy. Soc. Lond.*, A358:467–494, 1978.

[14] S. Hawking. Gravitational radiation from colliding black holes. *Phys. Rev. Lett.*, 26:1344–1346, 1971.

[15] S. Hawking. Black holes in general relativity. *Commun. Math. Phys.*, 25:152–166, 1972.

[16] S. Hawking. Particle creation by black holes. *Commun. Math. Phys.*, 43:199–220, 1975.

[17] S. Hawking and G. Ellis. *The large scale structure of space-time.* Cambridge University Press, 1973.

[18] G. Horowitz and R. Myers. AdS-CFT correspondence and a new positive energy conjecture for general relativity. *Phys. Rev. D*, 59:026005, 1999.

[19] D. Ida. Brane-world cosmology. *J. High Energy Phys.*, page JHEP09(2000)014, 2000.

[20] D. Ida. No black hole theorem in three-dimensional gravity. *Phys. Rev. Lett.*, 85:3758–3760, 2000.

[21] W. Jaco and P. Shalen. Seifert fibered spaces in 3-manifolds. *Mem. Am. Math. Soc.*, 21:1–192, 1979.

[22] K. Johannson. *Homotopy equivalences of 3-manifolds with boundary.* Number 761 in Lecture notes in mathematics. Springer-Verlag, 1979.

[23] H. Kneser. Geschlossene Flächen in dreidimensionale Mannigfaltigkeiten. *Jahresber. Deutsch. Math.-Verein.*, 38:248–260, 1929.

[24] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum. *Exact solutions of Einstein's field equations.* Cambridge University Press, 1980. Edited by E. Schmutzer.

[25] P. Kraus. Dynamics of anti-de Sitter domain walls. *J. High Energy Phys.*, page JHEP12(1999)011, 1999.

[26] P. Laplace. Le système du monde, volume 2. Paris, 1795.

[27] J. Lemos. Cylindrical black hole in general relativity. *Phys. Lett. B*, 352:46–51, 1995.

[28] J. Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[29] R. Mann. Pair production of topological anti-de Sitter black holes. *Class. Quant. Grav.*, 14:L109–L114, 1997.

[30] J. Milnor. A unique decomposition theorem for 3-manifolds. *Amer. J. Math.*, 84:1–7, 1962.

[31] C. Misner, K. Thorne, and J. Wheeler. *Gravitation.* W. H. Freeman and Company, 1973.

[32] B. O'Neill. *Semi-Riemannian geometry with applications to relativity.* Academic Press, 1983.

[33] P. Petersen. *Riemannian geometry.* Springer-Verlag, 1998.

[34] L. Randall and R. Sundrum. An alternative to compactification. *Phys. Rev. Lett.*, 83:4690–4693, 1999.

[35] M. Ryan, Jr., and L. Shepley. *Homogeneous relativistic cosmologies.* Princeton University Press, 1975.

[36] B. Schutz. *Geometrical methods of mathematical physics.* Cambridge University Press, 1980.

[37] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15:401–487, 1983.

[38] T. Shiromizu and D. Ida. Anti-de Sitter no-hair, AdS/CFT and the brane-world. `arXiv.org` e-Print hep-th/0102035, 2001. To appear in *Phys. Rev. D.*

[39] A. Taub. Empty space-times admitting a three-parameter group of motions. *Ann. Math.*, 53:472–490, 1951.

[40] W. Thurston. Three dimensional manifolds, Kleinian groups, and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6:357–381, 1982.

[41] W. Thurston. *Three dimensional geometry and topology*, volume 1. Princeton University Press, 1997.

[42] L. Vanzo. Black holes with unusual topology. *Phys. Rev. D*, 56:6475–6483, 1997.

[43] R. Wald. *General relativity.* University of Chicago Press, 1984.

[44] C. Will. *Theory and experiment in gravitational physics.* Cambridge University Press, revised edition, 1993.